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A Weak Galerkin Mixed Finite Element Method for Second Order Elliptic Equations on 2D Curved Domains

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Abstract. This article concerns the weak Galerkin mixed finite element method (WG-MFEM) for second order elliptic equations on 2D domains with curved boundary. The Neumann boundary condition is considered since it becomes the essential boundary condition in this case. It is well-known that the discrepancy between the curved physical domain and the polygonal approximation domain leads to a loss of accuracy for discretization with polynomial order $\alpha > 1$. The purpose of this paper is two-fold. First, we present a detailed error analysis of the original WG-MFEM for solving problems on curved domains, which exhibits an $O(h^{1/2})$ convergence for all $\alpha \geq 1$. It is a little surprising to see that even the lowest-order WG-MFEM ($\alpha = 1$) experiences a loss of accuracy. This is different from known results for the finite element method (FEM) or the mixed FEM, and appears to be a combined effect of the WG-MFEM design and the fact that the outward normal vector on the polygonal approximation domain is different from the one on the curved domain. Second, we propose a remedy to bring the approximation rate back to optimal by employing two techniques. One is a specially designed boundary correction technique. The other is to take full advantage of the nice feature that weak Galerkin discretization can be defined on polygonal meshes, which allows the curved boundary to be better approximated by multiple short edges without increasing the total number of mesh elements. Rigorous analysis shows that a combination of the above two techniques renders optimal convergence for all α. Numerical results further confirm this conclusion.

AMS subject classifications: 65N15, 65N30

Key words: Weak Galerkin method, polygonal mesh, curved domain, mixed formulation.

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1 Introduction

Many practical problems arising in science and engineering are posed on domains with curved boundaries. When such problems are approximated on polygonal or polyhedral computational domains, the geometric difference between the two leads to a loss of approximation accuracy [36, 37] for high-order elements. To resolve this issue, a straightforward idea is to reduce the geometric error down to the same level of the approximation error. Popular methods following this track include the isoparametric finite element method [22, 27] and the isogeometric analysis [21, 26]. However, due to their specialized design, neither of them can be applied to meshes consisting of polygons or polyhedra.

In the past two decades, discretizations on polygonal and polyhedral meshes have gained considerable attention in the scientific computing community. Various numerical schemes have been proposed, including the virtual element method (VEM) (see [3] and references therein), the discontinuous Galerkin method (DG) [19, 24, 33], and the weak Galerkin method (WG) [32, 40, 41], to name a few. Very recently, researchers start to apply these discretizations to curved domains, which requires innovative techniques, with its reason explained above. One emerging method is the boundary correction technique, which may have its root date back to a 1972 paper by Bramble, Dupont and Thomée [11]. The idea is to use normal-directional Taylor expansion, in most cases just a linear approximation, to correct function values on the boundary. Burman et al. [14, 15] proposed the technique for a CutFEM discretization in 2018. It was soon applied to VEM by Bertoluzza et al. [7]. In 2019, Cheung et al. [17] proposed a polynomial extension FEM which is based on an averaged Taylor expansion. In a series of papers starting from 2018, Main and Scovazzi [1, 28, 29] designed a shifted boundary FEM on non-fitted meshes, i.e., boundary nodes of the mesh may not lie on the curved physical boundary. However, their method uses 1st-order (linear) Taylor expansion and hence only works for linear elements. Finally, we mention an earlier but closely related work [20], where instead of Taylor expansion the authors used a path integration to achieve a similar 'boundary correction' effect.

A totally different track, first proposed for VEM by Beirão da Veiga et al. [5] in 2019, is to define the discretization directly on curved mesh elements. This is possible because of the 'skeletal' style design of VEM, where the degrees of freedom (dofs) in the interior and on the boundary of each mesh element are defined separately. In [5], dofs as well as related shape functions on curved boundary edges are defined using the parameter in the parametric equation of the curved boundary. Hence the shape functions on curved edges are no longer polynomials in the physical space. Later a modification was proposed [4] which uses the restriction of physical polynomials to define shape functions on curved boundary edges. In 2021, Mu applied the idea to the primal WG discretization [30]. Because of the direct use of the curved boundary, the implementation of these methods requires numerical integration formulae on curved mesh elements, as well as a mapping between each curved boundary segment and its flat counterpart in the parametric space. The theoretical analysis is also more complicated as it has to deal with the parametric