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# $\mathfrak{X}$-Gorenstein Projective Dimensions

Jie Wang[1], Xiaowei Xu[2] and Zhibing Zhao[3,*]

[1] *Department of Basic Teaching, Anhui Sanlian University, Hefei 230601, China;*
[2] *School of Mathematical Sciences, Jilin University, Changchun 130012, China;*
[3] *School of Mathematical Sciences, Anhui University, Hefei 230601, China.*

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**Abstract.** In this paper, we mainly investigate the $\mathfrak{X}$-Gorenstein projective dimension of modules and the (left) $\mathfrak{X}$-Gorenstein global dimension of rings. Some properties of $\mathfrak{X}$-Gorenstein projective dimensions are obtained. Furthermore, we prove that the (left) $\mathfrak{X}$-Gorenstein global dimension of a ring $R$ is equal to the supremum of the set of $\mathfrak{X}$-Gorenstein projective dimensions of all cyclic (left) $R$-modules. This result extends the well-known Auslander's theorem on the global dimension and its Gorenstein homological version.

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**Key words**: Gorenstein projective modules, $\mathfrak{X}$-Gorenstein projective modules, $\mathfrak{X}$-Gorenstein projective dimensions, the Auslander's theorem.

## 1 Introduction

Throughout this paper, $R$ always denotes a non-trivial associative ring with identity. All modules are unital left $R$-modules. We denote the category of all left $R$-modules by $R$-Mod. The full subcategory of $R$-Mod consisting of all projective $R$-modules (resp. all Gorenstein projective $R$-modules) are denoted by $\mathcal{P}(R)$ (resp. $\mathcal{GP}(R)$). We use $\mathrm{pd}(M)$ and $\mathrm{Gpd}(M)$ to denote, respectively, the projective dimension and Gorenstein projective dimension of $M$. We always assume that $\mathfrak{X}$ is a class of $R$-modules which contains all projective $R$-modules.

As a generalization of finitely generated projective modules over a noetherian ring $R$, Auslander and Bridger [3] introduced in 1969 the notion of the module with G-dimension zero. The G-dimension for finitely generated $R$-module $M$, denoted by G-dim$(M)$, is defined via resolution with this kind of module. They proved that G-dim$(M) \leq \mathrm{pd}(M)$,

*Corresponding author. *Email addresses:* `1570898331@qq.com` (Wang J), `xuxw@jlu.edu.cn` (Xu X), `zbzhao@ahu.edu.cn` (Zhao Z)

with equality G-dim$(M) = \text{pd}(M)$ when pd$(M)$ is finite. Furthermore, they give a characterization of the Gorenstein local rings in terms of G-dimension in [2]. Namely, $R$ is Gorenstein if its residue field $k$ has finite G-dimension, and only if every finitely generated $R$-module has finite G-dimension.

Over an arbitrary ring $R$, Enochs and Jenda defined in [11] the Gorenstein projective module as follows

**Definition.** *An R-module M is said to be Gorenstein projective if there exists a completed projective resolution*

$$\mathbf{P} := \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

*being exact in* $R$-Mod *which is still exact after applying* $\text{Hom}_R(-,P)$ *for any projective module* $P$ *and* $M = \text{Ker}(P_0 \to P^0)$.

*The exact sequence* $\mathbf{P}$ *is called a complete projective resolution of M.*

Avramov, Buchweitz, Martsinkovsky and Reiten proved that a finitely generated module over a noetherian ring is Gorenstein projective if and only if it is a module with G-dimension zero (see [8, Theorem 4.2.6] and [8, P.99]). The Gorenstein projective modules share many nice properties of the classical projective module (see, for instance, [7, 8, 12, 14, 15]). Gorenstein projective dimension of an arbitrary $R$-module, Gpd$(M)$, is defined via resolution with Gorenstein projective modules. This kind of homological dimension is analogous to the classical projective dimension and shares some of its principal properties (see [4, 5, 8, 14] for more details).

In [6], Bennis and Ouarghi introduced the notion of $\mathfrak{X}$-Gorenstein projective module where $\mathfrak{X}$ is a class of $R$-modules that contains all projective $R$-modules (see Definition 2.1). This notion uniforms some homological modules, including projective modules, Gorenstein projective modules and strongly Gorenstein flat modules and so on (see Remark 2.2). Similar to the Gorenstein projective dimension, the $\mathfrak{X}$-Gorenstein projective dimension is defined via resolution with $\mathfrak{X}$-Gorenstein projective modules.

In this paper, we will investigate some properties of $\mathfrak{X}$-Gorenstein projective dimension of modules and $\mathfrak{X}$-Gorenstein global projective dimensions of rings.

In Section 2, we will give some basic properties of $\mathfrak{X}$-Gorenstein projective modules.

Section 3 deals with the $\mathfrak{X}$-Gorenstein projective dimension, $\mathfrak{X}$-Gpd$(-)$. Firstly, we establish the following equality.

**Proposition.** *For any family of R-modules* $(M_i)_{i\in I}$,

$$\mathfrak{X}\text{-Gpd}(\bigoplus_{i\in I} M_i) = \sup\{\mathfrak{X}\text{-Gpd}(M_i) \mid i \in I\}.$$

Note that this result is a generalization of [14, Proposition 2.19]. The following theorem for $\mathfrak{X}$-Gorenstein projective dimension are similar to those of classical projective dimension.

**Theorem.** *Let* $0 \to A \to B \to C \to 0$ *be a short exact sequence in R-Mod. If two of the modules* $A, B$ *and* $C$ *have finite* $\mathfrak{X}$*-Gorenstein projective dimension, then so has the third.*