Journal of Computational Mathematics
Vol.41, No.1, 2023, 18–38.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2104-m2020-0250

RECONSTRUCTION OF SPARSE POLYNOMIALS VIA QUASI-ORTHOGONAL MATCHING PURSUIT METHOD*

Renzhong Feng and Aitong Huang[1)]
School of Mathematics Science & Key Laboratory of Mathematics, Informatics and Behavioral Semantics, Ministry of Education, Beihang University, Beijing 100191, China
Email: fengrz@buaa.edu.cn, huangaitong@buaa.edu.cn
Ming-Jun Lai and Zhaiming Shen
Department of Mathematics, University of Georgia, Athens, GA 30602, U.S.A.
Email: mjlai@uga.edu, Zhaiming.Shen@uga.edu

Abstract

In this paper, we propose a Quasi-Orthogonal Matching Pursuit (QOMP) algorithm for constructing a sparse approximation of functions in terms of expansion by orthonormal polynomials. For the two kinds of sampled data, data with noises and without noises, we apply the mutual coherence of measurement matrix to establish the convergence of the QOMP algorithm which can reconstruct s-sparse Legendre polynomials, Chebyshev polynomials and trigonometric polynomials in s step iterations. The results are also extended to general bounded orthogonal system including tensor product of these three univariate orthogonal polynomials. Finally, numerical experiments will be presented to verify the effectiveness of the QOMP method.

Mathematics subject classification: 41A10, 41A05, 65D15.
Key words: Reconstruction of sparse polynomial, Compressive sensing, Mutual coherence, Quasi-orthogonal matching pursuit algorithm.

1. Introduction

Since [11], it has been an interesting research topic to accurately reconstruct functions via a sparse representation with respect to an orthogonal basis. Suppose that $\Omega \subseteq \mathbb{R}^d$ and that $\{\phi_j(x)\}_{j\in\Lambda}$ is a set of orthogonal polynomials defined on Ω, where Λ is an index set. Suppose that $|\Lambda| = n$, here $|\cdot|$ means the cardinality of Λ, and $n \gg 1$ can be finite or infinite. For any continuous function g on Ω, we can have an orthonormal expansion of g:

$$g(x) = \sum_{j\in\Lambda} c_j\phi_j(x). \tag{1.1}$$

In any practical computation, one can not have a memory to store all the coefficients c_j. Thus one is interested in finding a sparse representation of g in the sense that

$$g(x) \approx \sum_{j\in\Lambda_s} \tilde{c}_j\phi_j(x) \tag{1.2}$$

for a given integer $s \geq 1$, where $\Lambda_s \subset \Lambda$ and $|\Lambda_s| = s \ll n$. That is, letting $\tilde{\mathbf{c}}$ be all the coefficient vectors containing all nonzero coefficients in (1.2), if $|\Lambda_s| \ll n$ and $\approx$ is $=$, the right-hand side

* Received September 21, 2020 / Revised version received February 26, 2021 / Accepted April 26, 2021 / Published online August 16, 2021 /
1) corresponding author

of the expression in (1.2) is said to be a sparse representation of g. For practical purpose, we say the right-hand side of the expression in (1.2) is a sparse approximation of g. The problem of finding $\tilde{\mathbf{c}}$ is naturally translated into reconstructing the s-sparse vector $\tilde{\mathbf{c}}$.

Since [1, 7, 8, 14], compressed sensing has been a popular research topic. Its main idea is to use the sparsity of signal to reconstruct the signal by using as few observations as possible. The original model of compressed sensing is

$$\min_{\mathbf{c}\in\mathbb{C}^n} ||\mathbf{c}||_0 \quad s.t. \quad \Phi\mathbf{c} = \mathbf{b}, \tag{1.3}$$

where $||\mathbf{c}||_0$ represents the number of non-zero elements in the vector $\mathbf{c}$, that is, the sparsity of the vector $\mathbf{c}$, $\Phi \in \mathbb{C}^{m\times n}$ is a measurement matrix or sensing matrix, and $\mathbf{b} \in \mathbb{C}^{m\times 1}$ is an observation vector, such as $b_i = g(x_i), i = 1, \cdots, m$ for some locations $x_i \in \Omega$. The model (1.3) is the model of the sampled data without noise. Since there may be noise in sampled data, assume that the noise bound is ε, then the model (1.3) can be written as follows

$$\min_{\mathbf{c}\in\mathbb{C}^n} ||\mathbf{c}||_0 \quad s.t. \quad ||\Phi\mathbf{c} - \mathbf{b}||_2 \le \varepsilon, \tag{1.4}$$

where $||\cdot||_2$ represents Euclidean norm. If we know in advance that the sparsity of the vector to be restored is s, then the problem (1.4) can be rewritten as

$$\min_{\mathbf{c}\in\mathbb{C}^n} ||\Phi\mathbf{c} - \mathbf{b}||_2 \quad s.t. \quad ||\mathbf{c}||_0 \le s. \tag{1.5}$$

Many researchers have applied the technique of solving the sparse signal in compressed sensing to reconstruct the coefficient vector $\tilde{\mathbf{c}}$, see, e.g. [21], [19], [20], [23] and [10].

A greedy algorithm is one of the common methods to solve the problem (1.5). Among them, the orthogonal matching pursuit algorithm (OMP for short) proposed in [6] is an important one of the greedy algorithms. Currently, there have been many improvements of the OMP method, such as regularized Orthogonal Matching Pursuit [12], Generalized Orthogonal Matching Pursuit [15], stagewise Orthogonal Matching Pursuit [16], etc. In this paper we propose an improved method of OMP method named Quasi-Orthogonal Matching Pursuit (QOMP for short). Different from the traditional OMP method, the QOMP method selects the two columns that are most related to the space of the current redundant vector expansion in each iteration.

Algorithm 1.1. Quasi-Orthogonal Matching Pursuit (QOMP)

Input: $\Phi_{m\times n}, \mathbf{b}_{m\times 1}$, sparsity s, maximum number of iterations $k_{\max}(k_{\max} < m/2)$, tolerance ε

Initialization: $S_0 = \emptyset$, $\mathbf{r}_0 = (b), k = 0, \Psi_{m\times n} = \Phi_{m\times n}$

while $k < k_{\max}$ and $||\mathbf{r}_k||_2 > \varepsilon$

$k = k + 1$

$Res_{(i,j)}(\mathbf{r}_{k-1}) = \min_{u,v\in\mathbb{R}}\{||\Psi_i u + \Psi_j v - \mathbf{r}_{k-1}||_2\}$

$(i_k, j_k) = \operatorname{argmin}_{1\le i,j\le n}\{Res_{(i,j)}(\mathbf{r}_{k-1})\}$

$S_k = S_{k-1} \cup \{i_k, j_k\}$

$\mathbf{r}_k = \mathbf{b} - \Phi_{S_k}\Phi_{S_k}^{\dagger}\mathbf{b}$

$\Psi_{i_k,j_k} = \mathbf{0}$

end while

Output:$S = S_k, \mathbf{c}_S = \Phi_S^{\dagger}\mathbf{b}$ and $\mathbf{c}_{S^c} = \mathbf{0}$