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# EXPONENTIAL TIME DIFFERENCING-PADÉ FINITE ELEMENT METHOD FOR NONLINEAR CONVECTION-DIFFUSION-REACTION EQUATIONS WITH TIME CONSTANT DELAY*

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**Abstract**

In this paper, ETD3-Padé and ETD4-Padé Galerkin finite element methods are proposed and analyzed for nonlinear delayed convection-diffusion-reaction equations with Dirichlet boundary conditions. An ETD-based RK is used for time integration of the corresponding equation. To overcome a well-known difficulty of numerical instability associated with the computation of the exponential operator, the Padé approach is used for such an exponential operator approximation, which in turn leads to the corresponding ETD-Padé schemes. An unconditional $L^2$ numerical stability is proved for the proposed numerical schemes, under a global Lipshitz continuity assumption. In addition, optimal rate error estimates are provided, which gives the convergence order of $O(k^3+h^r)$ (ETD3-Padé) or $O(k^4+h^r)$ (ETD4-Padé) in the $L^2$ norm, respectively. Numerical experiments are presented to demonstrate the robustness of the proposed numerical schemes.

*Mathematics subject classification:* 65N08, 65N12, 65N15.
*Key words:* Nonlinear delayed convection diffusion reaction equations, ETD-Padé scheme, Lipshitz continuity, $L^2$ stability analysis, Convergence analysis and error estimate.

## 1. Introduction

A delayed partial differential equation (DPDE) is formulated as

$$u_t + Au = f(t, u(\mathbf{x}, t), u(\mathbf{x}, t-\tau), \nabla u(\mathbf{x}, t), \nabla u(\mathbf{x}, t-\tau)), \tag{1.1}$$

in which $\tau$ is a fixed time quantity and $A$ is a linear operator. This equation has played an important role in the simulation of many real-world problems, such as biological systems [1], epidemiology [32, 36], medicine, engineering control systems, climate models [9], etc. A variety of phenomenon in the natural and social sciences could be described by such a model, so that there have always been great significance and practical values to study it.

One distinguished feature of this model is associated with the fact that, the unknown quantity $u(\mathbf{x}, t)$ depends not only on the solution at the present stage, but also on the solution at some past stage. For a survey of early results, we refer the readers to [39] and references therein.

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Since the 1950s, theoretical research of delayed differential equations has been attracted more and more attentions, with very rapid developments. The theoretical results on delay differential equations have already been quite rich [4, 19, 20, 22], while that for delayed partial differential equation is still an active developing branch [17,34,40,41,43]. In [34] the global asymptotic stability of traveling waves was studied for delayed reaction-diffusion equations; a similar analysis was carried out in [41] for Nicholson's blowflies equation with diffusion; more stability analysis have also been presented in [43]. A necessary and sufficient condition for oscillatory behavior of neutral hyperbolic delay partial differential equations was provided in [40]; the delayed reaction-diffusion model of the bacteriophage infection spread of bacteriophage infection was studied by Gourley and Kuang [17].

On the other hand, it is well-known that, an analytical solution for most DPDEs is not available [39]; even for some simple and specific equations, the theoretical solutions are often piecewise continuous or even cannot be expressed in analytical expression. In turn, the numerical investigation on DPDEs has attracted more attentions. For example, the Chebyshev spectral collocation method with waveform relaxation was considered in [21] for nonlinear D-PDEs. A local discontinuous Galerkin (LDG) method was proposed in [30] for reaction-diffusion dynamical systems with time delay. Also see [31, 33] and the related references.

Meanwhile, most existing numerical works have been focused on either first or second order accurate (in time) schemes; the numerical study of temporally third order or even higher-order numerical schemes for the delayed equation (1.1), as well as its theoretical analysis, has been very limited. In this article, we propose and analyze (temporally) third order and fourth order accurate numerical schemes for the delayed convection-diffusion-reaction equation, based on the exponential time differencing (ETD) temporal algorithm, combined with finite element spatial approximation. In general, an exact integration of the linear part of the PDE is involved in the ETD-based scheme. For the nonlinear terms, the multi-step approach has been based on explicit approximation of the temporal integral [2, 11, 18]. An application of such an idea to various gradient models has been reported in recent works [5–7,12,13,23–25,38,46]. On the other hand, in the extension of these ETD-based ideas to the delayed PDE, a higher order interpolation has to be involved in the numerical evaluation of the physical variables at the staggered temporal stencil mesh point around the delayed time instant. In addition, a direct computation of the exponential operator may lead to numerical instability, as discussed in an earlier work [26]. To overcome this well-known difficulty, the Padé approach is used for such an exponential operator approximation, which in turn leads to the corresponding ETD-Padé scheme, with the third or fourth order accuracy in the time discretization. The spatial discretization is the same as the ones to deal with the non-delay problem. In addition, the ETD-Padé schemes can reach the corresponding convergence order as the ETDRK schemes and the computational cost is less, also see the related derivations in [14, 27], etc.

A theoretical analysis of the third and fourth order accurate ETD-Padé schemes turns out to be highly challenging, due to the nonlinear and multi-step nature, combined with the delayed structure. An unconditional $L^2$ numerical stability is proved for the proposed schemes, under a global Lipshitz continuity assumption. With that assumption, the nonlinear difference could bounded by a linear growth term. In combination of the eigenvalue estimates for the operators involved in the ETD-Padé scheme, the $L^2$ numerical stability is derived with the help of discrete Gronwall inequality. In addition, an optimal rate convergence analysis and error estimate could be established in the same manner, in which the stability of the difference between the exact solution and numerical solution is analyzed. This in turn gives a convergence order of $O(\tau^3+h^r)$