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# POSITIVITY-PRESERVING LOCAL DISCONTINUOUS GALERKIN METHOD FOR PATTERN FORMATION DYNAMICAL MODEL IN POLYMERIZING ACTIN FLOCKS*

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**Abstract**

In this paper, we apply local discontinuous Galerkin (LDG) methods for pattern formation dynamical model in polymerizing actin flocks. There are two main difficulties in designing effective numerical solvers. First of all, the density function is non-negative, and zero is an unstable equilibrium solution. Therefore, negative density values may yield blow-up solutions. To obtain positive numerical approximations, we apply the positivity-preserving (PP) techniques. Secondly, the model may contain stiff source. The most commonly used time integration for the PP technique is the strong-stability-preserving Runge-Kutta method. However, for problems with stiff source, such time discretizations may require strictly limited time step sizes, leading to large computational cost. Moreover, the stiff source any trigger spurious filament polarization, leading to wrong numerical approximations on coarse meshes. In this paper, we combine the PP LDG methods with the semi-implicit Runge-Kutta methods. Numerical experiments demonstrate that the proposed method can yield accurate numerical approximations with relatively large time steps.

*Mathematics subject classification:* 65M15, 65M60.
*Key words:* Pattern formation dynamical model, Local discontinuous Galerkin method, Positive-preserving technique, Semi-implicit Runge-Kutta method, Stiff source.

## 1. Introduction

The actin, first discovered by Halliburton in 1887, forms cable-like structures called filaments that make up most of the cell's supporting skeleton. Actin is a kind of active polymer. Its polymerization and depolymerization process play an important role in cell activities, such as cell division and movement, intercellular communication, etc. In recent days, the mathematical model of the pattern wave for the actin filaments has been investigated in [5, 9, 10]. Generally speaking, it is divided into two steps in the process of wave formation. Small actin filaments first aggregate into spots or spirals, and then evolve into waveforms [9, 16].

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The pattern formation dynamical model in polymerizing actin flocks reads [9,16]:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = -v_0 \nabla \cdot (\rho \mathbf{P}) + D_\rho \nabla^2 \rho + \alpha \rho \left(1 - \dfrac{\rho}{\rho_0}\right), & (x,y) \in \Omega,\ 0 < t \le T, \\ \dfrac{\partial \mathbf{P}}{\partial t} = \gamma \left(\dfrac{\rho}{\rho_c} - 1\right) \mathbf{P} + K \nabla^2 \mathbf{P} - \gamma_2 |\mathbf{P}|^2 \mathbf{P}, & (x,y) \in \Omega,\ 0 < t \le T, \end{cases}$$

where $\Omega$ is a rectangular domain in $R^2$. $\rho$ and $\mathbf{P}$ denote the density and average filament polarization of F-actin, respectively. $v_0$ is the treadmilling speed and $\alpha$ is the polymerization rate. $D_\rho$ and $K$ are the diffusion coefficient and the effective elastic constant, respectively. $\gamma$ denotes the rate of change of the direction of F-actin filaments and $\gamma_2$ guarantees saturation of the polarization. $\rho_c$ and $\rho_0$ denote the critical density and the target polymerization density, respectively. To facilitate analysis, we replace $\alpha t$, $(\alpha/D_\rho)^{1/2} x$, $\rho/\rho_0$, $(v_0/\sqrt{D_\rho \alpha})\mathbf{P}$ with $t$, $x$, $\rho$, $\mathbf{P}$ respectively, so we can rewrite the model as follows:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{P}) + \nabla^2 \rho + \rho (1 - \rho), \qquad (x,y) \in \Omega,\ 0 < t \le T, \tag{1.1a}$$

$$\frac{\partial \mathbf{P}}{\partial t} = \Gamma (r\rho - 1) \mathbf{P} + D \nabla^2 \mathbf{P} - \Gamma_2 |\mathbf{P}|^2 \mathbf{P}, \qquad (x,y) \in \Omega,\ 0 < t \le T, \tag{1.1b}$$

where $\Gamma = \gamma/\alpha$, $D = K/D_\rho$, $\Gamma_2 = \gamma_2 D_\rho / v_0^2$, $r = \rho_0/\rho_c$, $\mathbf{P} = (p_1, p_2)$, and $|\mathbf{P}|^2 = p_1^2 + p_2^2$. Moreover, for simplicity of presentation, we consider periodic boundary conditions in this paper. The initial conditions are given as

$$\rho(x,y,0) = \rho^0(x,y), \quad p_1(x,y,0) = p_1^0(x,y), \quad p_2(x,y,0) = p_2^0(x,y). \tag{1.2}$$

There are not too many works discussing numerical methods for the pattern formation dynamical model. The first work in this direction was given in [9], where the authors applied finite difference methods to simulate the pattern formation. Later, the characteristic finite element analysis was introduced in [16]. To capture the detailed structures, such as spots, spirals and waves, we constructed the local discontinuous Galerkin (LDG) methods for pattern formation dynamical model and obtained the optimal error estimates in [23]. The LDG method, as an extension of the discontinuous Galerkin (DG) method [20], was first introduced in [4] for convection-diffusion equations, motivated by the work given by Bassi and Rebay [1]. The main idea is to rewrite the equation with higher order derivatives into a first order system by introducing new auxiliary variables, and then apply the DG method to the system. The LDG method shares the same advantages of the DG method, such as good stability, high order accuracy, and flexibility on h-p adaptivity.

Though the contributions given in [9,16,23] work for most of the regular cases, they may fail to work in some special cases. In fact, there are two main difficulties in designing robust numerical algorithms.

1. The density in (1.1a) is non-negative and zero is an unstable equilibrium. If the density in (1.1a) is negative, then the source of (1.1a) is negative, which causes the density to deviate from zero. Numerical experiments in Section 5 demonstrate that negative numerical densities may yield blow-up numerical approximations. To fix this gap, we will apply the positivity-preserving (PP) technique. In 2010, Zhang and Shu [29] first constructed the genuinely maximum-principle-preserving (MPP) high-order DG shemes for conservation laws on rectangular meshes. Later, the technique was successfully extended to problems