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A PRIORI ERROR ESTIMATES FOR OBSTACLE OPTIMAL CONTROL PROBLEM, WHERE THE OBSTACLE IS THE CONTROL ITSELF*

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Abstract

In this paper we deal with the convergence analysis of the finite element method for an elliptic penalized unilateral obstacle optimal control problem where the control and the obstacle coincide. Error estimates are established for both state and control variables. We apply a fixed point type iteration method to solve the discretized problem.

To corroborate our error estimations and the efficiency of our algorithms, the convergence results and numerical experiments are illustrated by concrete examples.

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Key words: Optimal control, Obstacle problem, Finite element, A priori error estimate.

1. Introduction

The field of variational inequalities and related optimal control problems (when the obstacle is a given function different from the control function) with the control variables appearing in the variational inequality have been extensively studied in the literature with different formulations, see for example [14, 20–22, 25, 29] and the references therein.

Another class of this type of problems is when the obstacle functions are unknowns and are considered as control functions. These problems are encountered, for example, in shape optimisation [7, 9]. It may also concern a dam optimal shape see, for instance [6], where the obstacle gives the shape to be designed such that the pressure of the fluid inside the dam is close to a desired value. This is equivalent in some sense to controlling the free boundary, see [18]. Moreover, these problems may find applications in a large area of disciplines, for example, physics, engineering, economics and biology, see [5, 10]. Different approaches have been used to study this kind of problems [13], for example in [19, 29], the authors used penalized techniques to approximate the variational inequality which leads to optimal control problems governed by variational equations which depend on a penalized approximation parameter.

To our knowledge, there have been extensive theoretical and numerical studies for the finite element discretization of various optimal control problems. In particular, a priori error estimates for finite element discretization of optimal control problems governed by elliptic equations have been established for example in [2, 21, 23, 26, 28, 31].

However, there are only a few publications dealing with a priori error estimates for problems governed by variational inequalities and obstacle problem of this type, see [8, 11, 24].

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Nevertheless, the optimal control of variational inequalities of obstacle type is still a very active field of research especially for their numerical treatment. In the mathematical literature, there are few works on the numerical resolution of this kind of problems (when the control and the obstacle are the same). Ghanem et al. in [16] and Kunish in [19] are, to our knowledge, the only ones treating numerically this kind of problems by using finite difference method for only unilateral obstacle optimal control.

The main purpose of this paper is twofold. First, by using the same ideas and techniques given in [16], we resolve numerically the penalized unilateral obstacle optimal control problem (given below by (P_h^δ)) by using the indirect approach (after optimisation). We first begin by discretizing the optimality system (2.5) by using the finite element method and with a simple iterative algorithm based on Gauss-Seidel method which combines Newton-Raphson and a direct method. The proposed algorithm is designed to have a fixed point.

As it is known, on one side, the solution of the state equation of such problems, nonlinearly depends on the control variable [19] and, on the other, owing to the absence of convexity, we cannot guarantee the uniqueness of the optimal solution of this class of problems. This is why we focus on locally optimal controls.

Secondly, we establish the an priori error estimate $\left\|\overline{\varphi}^\delta - \overline{\varphi}_h^\delta\right\|_{H_0^1(\Omega)}$ with respect to the discretization parameter h, where $\overline{\varphi}_h^\delta$ is an optimal control solution of the discrete problem (P_h^δ), and $\overline{\varphi}^\delta$ is a locally optimal control of (P^δ).

Afterward, we derive the convergence rate and justify the independence of the rate of convergence with respect to the penalized parameter.

Furthermore, we note that, the principal difficulty in this work, is to derive the priori error estimate of the control by using the projection equation which is a nonlinear and anti-monotone variational inequality.

In order to avoid cumbersome notation, C and c denote generic positives constants independent of the approximation parameter h and eventually depending on the penalized parameter δ, which may not be the same at each occurrence and $B_V(u, r)$ denote the V-ball of radius r around u.

The remainder of this paper is organized as follows. In the next Section, we begin with a general introduction. Then, in Section 3, we shall give a brief review on the finite element method and then construct the finite element approximation for the penalized control problem, convergence of the discrete problem is discussed and a priori error bounds are derived.

To validate our theoretical results some numerical examples and iterative algorithm are given in Section 4. At the end of this paper, we draw conclusions.

2. Preliminaries and Known Results

In this section, we begin by giving some material that will be used in our subsequent analysis. Throughout the paper, Ω is an open bounded set in $\mathbb{R}^n$ $(n \leq 3)$, with a boundary $\partial\Omega$ of class $\mathscr{C}^2$. For $1 \leq p \leq +\infty$, we adopt the standard notation for the Sobolev space $W^{m,p}(\Omega)$ (see for example [1]), where

$$W^{m,p}(\Omega) = \{v \mid v \in L^p(\Omega) \mid \forall\alpha \text{ in } \mathbb{N}^n, |\alpha| \leq m,\ \partial^\alpha v \in L^p(\Omega)\},$$

and the positive integer m is called the order of the Sobolev space $W^{m,p}(\Omega)$. For the special case, where $p = 2$, we denote by $H^m(\Omega)$ the Sobolev space of order $m > 0$ in Ω with norm