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Hyperbolic Divergence Cleaning in Lattice Boltzmann Magnetohydrodynamics

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Abstract. Magnetohydrodynamics couples the Navier–Stokes and Maxwell's equations to describe the flow of electrically conducting fluids in magnetic fields. Maxwell's equations require the divergence of the magnetic field to vanish, but this condition is typically not preserved exactly by numerical algorithms. Solutions can develop artifacts because structural properties of the magnetohydrodynamic equations then fail to hold. Magnetohydrodynamics with hyperbolic divergence cleaning permits a nonzero divergence that evolves under a telegraph equation, designed to both damp the divergence, and propagate it away from any sources, such as poorly resolved regions with large spatial gradients, without significantly increasing the computational cost. We show that existing lattice Boltzmann algorithms for magnetohydrodynamics already incorporate hyperbolic divergence cleaning, though they typically use parameter values for which it reduces to parabolic divergence cleaning under a slowly-varying approximation. We recover hyperbolic divergence cleaning by adjusting the relaxation rate for the trace of the tensor that represents the electric field, and absorb the contribution from the symmetric-traceless part of this tensor using a change of variables. Numerical experiments confirm that the qualitative behaviour changes from parabolic to hyperbolic when the relaxation time for the trace of the electric field tensor is increased.

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1 Introduction

Magnetohydrodynamics (MHD) describes flows of electrically conducting fluids in magnetic fields by coupling the Navier–Stokes equations with Maxwell's equations. The latter require the magnetic field to have zero divergence, but this condition is typically not

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preserved exactly in numerical simulations. A non-vanishing $\nabla\cdot\mathbf{B}$ can create artifacts in simulations because structural properties of the MHD equations fail to hold [1–4]. For example, the divergence of the Maxwell stress $\frac{1}{2}|\mathbf{B}|^2\mathbf{I}-\mathbf{BB}$ is no longer equal to minus the Lorentz force $(\nabla\times\mathbf{B})\times\mathbf{B}$, and no longer perpendicular to the magnetic field $\mathbf{B}$.

There are several approaches to resolving this problem, some inspired by earlier work on an analogous problem for the electric field $\mathbf{E}$ in electrostatically interacting systems. While $\nabla\cdot\mathbf{E}$ is generally not zero in Maxwell's equations, a consistency condition connects the evolution of $\nabla\cdot\mathbf{E}$ with the electric current $\mathbf{J}_e$. This consistency condition is usually not preserved by numerical algorithms, especially those using the "particle-in-cell" approach [5–8].

Yee's [9] finite difference time domain (FDTD) scheme for Maxwell's equations exactly preserves a particular discrete approximation to $\nabla\cdot\mathbf{B}=0$ by representing the electric and magnetic fields on a staggered grid. Evans & Hawley [10] extended this scheme to MHD with a form of artificial viscosity that they named constrained transport. DeVore [11] designed a flux corrected transport scheme with the same property, and with flux limiters to resolve discontinuous solutions of the ideal MHD equations. Tóth [3] showed that these schemes can be transformed into standard finite volume schemes on unstaggered grids. All these schemes are ancestors of more recent mimetic discretisations that ensure that the vector identity $\widetilde{\nabla}\cdot(\widetilde{\nabla}\times(\cdots))=0$ holds for consistent discrete divergence $\widetilde{\nabla}\cdot(\cdots)$ and curl $\widetilde{\nabla}\times(\cdots)$ operators [12,13].

Brackbill & Barnes [1], and also Boris [5], proposed a projection method for evolving the magnetic field in discrete timesteps of length Δt, following the pressure projection method used for solving the incompressible Navier–Stokes equations [14–16]. In its simplest form, this projection method is:

$$\mathbf{B}^\star=\mathbf{B}^n-\Delta t(\widetilde{\nabla}\times\mathbf{E})^n, \tag{1.1a}$$

$$\mathbf{B}^{n+1}=\mathbf{B}^\star-\widetilde{\nabla}\Psi^{n+1}, \tag{1.1b}$$

where $\widetilde{\nabla}$ denotes a consistent discrete approximation to the gradient operator. The magnetic field $\mathbf{B}^n$ is evolved forwards by a single timestep to define an intermediate solution $\mathbf{B}^\star$. This intermediate solution is then projected onto the space of divergence-free vector fields by subtracting the gradient of a scalar field Ψ^{n+1} determined by solving the elliptic equation $\widetilde{\nabla}^2\Psi^{n+1}=\widetilde{\nabla}\cdot\mathbf{B}^\star$. This method ensures that $\widetilde{\nabla}\cdot\mathbf{B}^{n+1}=0$ provided the various discrete operators satisfy $\widetilde{\nabla}^2\Psi=\widetilde{\nabla}\cdot(\widetilde{\nabla}\Psi)$ [2,3,17].

Dedner *et al.* [18] considered a set of MHD equations based on an extended form of Maxwell's equations with [6–8]

$$\partial_t\mathbf{B}+\nabla\times\mathbf{E}+\nabla\Psi=0, \tag{1.2a}$$

$$\mathcal{D}(\Psi)+\nabla\cdot\mathbf{B}=0. \tag{1.2b}$$

The evolution equation for $\mathbf{B}$ contains an extra contribution from the gradient of a scalar field Ψ that is related to $\nabla\cdot\mathbf{B}$ by a general linear operator $\mathcal{D}$. We can interpret the projection method (1.1a,b) as a particular discretisation of (1.2a,b) with $\mathcal{D}(\Psi)=-\nabla^2\Psi$ via