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Weak Solutions to the Equations of Stationary Compressible Flows in Active Liquid Crystals

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Abstract. The equations of stationary compressible flows of active liquid crystals are considered in a bounded three-dimensional domain. The system consists of the stationary Navier-Stokes equations coupled with the equation of Q-tensors and the equation of the active particles. The existence of weak solutions to the stationary problem is established through a two-level approximation scheme, compactness estimates and weak convergence arguments. Novel techniques are developed to overcome the difficulties due to the lower regularity of stationary solutions, a Moser-type iteration is used to deal with the strong coupling of active particles and fluids, and some weighted estimates on the energy functions are achieved so that the weak solutions can be constructed for all values of the adiabatic exponent $\gamma > 1$.

AMS subject classifications: 35Q35, 35Q30, 35D35, 76D05, 76A15

Key words: Active liquid crystals, stationary compressible flows, Navier-Stokes equations, Q-tensor, weak solutions, weak convergence.

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1 Introduction

Active hydrodynamics refer to dynamical systems that are continuously driven out of equilibrium state by injected energy effects on small scales and exhibit collective phenomenon on a large scale, for example, bacterial colonies, motor proteins, and living cells [2, 24, 25]. Active systems have natural analogies with nematic liquid crystals because the particles exhibit a orientational ordering at a high concentration due to the collective motion. In comparison with the passive nematic liquid crystals, the system of active hydrodynamics is usually unstable and has novel characteristics such as low Reynolds numbers and very different spatial and temporal patterns [15,34]. We refer the readers to [5,15,18,25,29,33,34] and their references for the physical background, applications and modeling of active hydrodynamics. Theoretical studies on active liquid crystals are relatively new and have attracted a lot of attention in recent years. For example, the evolutionary incompressible flows of active liquid crystals were studied in [6, 19] and the evolutionary compressible flows were investigated in [7, 32]. In this paper we are concerned with the stationary compressible flows of active liquid crystals, described by the following equations in a bounded domain $\mathcal{O}\subset\mathbb{R}^3$:

$$\begin{cases}\operatorname{div}(\rho\mathbf{u})=0, & (1.1a)\\ \mathbf{u}\cdot\nabla c-\triangle c=g_1, & (1.1b)\\ \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})+\nabla\rho^\gamma-\operatorname{div}\big(\mathbb{S}_{ns}(\nabla\mathbf{u})+\mathbb{S}_1(Q)+\mathbb{S}_2(c,Q)\big)=\rho g_2, & (1.1c)\\ \mathbf{u}\cdot\nabla Q+Q\Omega-\Omega Q+c_*Q\operatorname{tr}(Q^2)+\dfrac{(c-c_*)}{2}Q & \\ \quad -b\left(Q^2-\dfrac{1}{3}\operatorname{tr}(Q^2)\mathbb{I}\right)-\triangle Q=g_3, & (1.1d)\end{cases}$$

where $\rho, c, \mathbf{u}$ denote the total density, the concentration of active particles, and the velocity field, respectively; the nematic tensor order parameter Q is a traceless and symmetric 3×3 matrix, ρ^γ is the pressure with adiabatic exponent $\gamma>1$, and the functions g_i, $i=1,2,3$ are given external force terms. We denote the Navier-Stokes stress tensor by

$$\mathbb{S}_{ns}(\nabla\mathbf{u})=\mu\big(\nabla\mathbf{u}+(\nabla\mathbf{u})^\top\big)+\lambda\operatorname{div}\mathbf{u}\mathbb{I}, \tag{1.2}$$

where $(\nabla\mathbf{u})^\top$ denotes the transpose of $\nabla\mathbf{u}$, $\mathbb{I}$ is the identity matrix, and the constants μ,λ are viscous coefficients satisfying the following physical requirement:

$$\mu>0,\quad \mu+3\lambda\geq 0. \tag{1.3}$$