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Convergence Rate of Solutions to a Hyperbolic Equation with $p(x)$-Laplacian Operator and Non-Autonomous Damping

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Abstract. This paper concerns the convergence rate of solutions to a hyperbolic equation with $p(x)$-Laplacian operator and non-autonomous damping. We apply the Faedo-Galerkin method to establish the existence of global solutions, and then use some ideas from the study of second order dynamical system to get the strong convergence relationship between the global solutions and the steady solution. Some differential inequality arguments and a new Lyapunov functional are proved to show the explicit convergence rate of the trajectories.

AMS subject classifications: 35L20

Key words: Convergence rate, energy estimate, non-autonomous damping.

1 Introduction

It is well-known that quasilinear wave equations involving p-Laplacian operator and damping term may describe the longitudinal motion of a rod made from a viscoelastic material [16]. In particular, the so-called Ludwick materials obey the following differential equations under the effect of an external force f, for the Euler rod:

$$\rho\frac{\partial^2 u}{\partial t^2}=K\frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{n-1}\frac{\partial u}{\partial x}\right)+f,$$

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and for the Euler beam

$$\rho A\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2}{\partial x^2}\left(KI_n\left|\frac{\partial^2 u}{\partial x^2}\right|^{n-1}\frac{\partial^2 u}{\partial x^2}\right)+f,$$

where ρ is the density of the material, A is the cross-sectional area, K is the material constant, n is the strain-hardening exponent, I_n is the second moment of inertia of the cross-section for the material, and $u=u(x,t)$ is the displacement at the time t and the space coordinate x. For these equations, there are many research achievements such as local existence, existence or non-existence of global solutions, asymptotic behavior as well as global attractor. We refer to [2, 4, 5, 12–14, 17, 18] and the references therein. These equations cannot describe accurately the motions of some fluids such as viscoelastic fluids, electrorheological fluids, processes of filtration through a porous media, fluids with temperature-dependent viscosity [1,6]. Stavrakakis and Stylianou [15] proposed the following model:

$$\begin{cases}\varepsilon u_{tt}-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)-\Delta u_t+g(u)=f(x,t), & (x,t)\in\Omega\times(0,T),\\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), & x\in\Omega,\end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N(N\geqslant 1)$ is a bounded domain with Lipschitz continuous boundary $\partial\Omega$. The existence and nonexistence of solutions to problem (1.1) were proved. It was shown that the behavior of solutions becomes more complicated if there are both damping and source. In fact, the damping drives the equation towards stability, while the source makes the equation unstable. For example, Antontsev [3] discussed the blowing-up properties of the corresponding problem for a negative initial energy. Later, Guo and Gao [8] improved these results. In addition, Messaoudi and Talahmeh [11] applied energy estimate method and differential inequality argument to discuss the blow-up behavior of solutions to the generalized telegraph equation with nonlinear source

$$\begin{cases}\varepsilon u_{tt}-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)+\mu u_t=|u|^{m(x)-2}u, & (x,t)\in\Omega\times(0,T),\\ u(x,t)=0, & (x,t)=\partial\Omega\times(0,T),\\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), & x\in\Omega,\end{cases}$$

where $\mu>0$. Motivated by the works above, we consider the following problem involving rotational inertial and mixed-type damping:

$$\begin{aligned}&u_{tt}-\Delta u_{tt}-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)+a(t)u_t-b\Delta u_t+g(u)\\ &=f(x,t),\quad (x,t)\in\Omega\times(0,T),\end{aligned} \tag{1.2a}$$