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Convergence of Weak Galerkin Finite Element Method for Second Order Linear Wave Equation in Heterogeneous Media

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Abstract. Weak Galerkin finite element method is introduced for solving wave equation with interface on weak Galerkin finite element space $(\mathcal{P}_k(K), \mathcal{P}_{k-1}(\partial K), [\mathcal{P}_{k-1}(K)]^2)$. Optimal order a priori error estimates for both space-discrete scheme and implicit fully discrete scheme are derived in $L^\infty(L^2)$ norm. This method uses totally discontinuous functions in approximation space and allows the usage of finite element partitions consisting of general polygonal meshes. Finite element algorithm presented here can contribute to a variety of hyperbolic problems where physical domain consists of heterogeneous media.

AMS subject classifications: 65M15, 65M60

Key words: Wave equation, heterogeneous medium, finite element method, weak Galerkin method, semidiscrete and fully discrete schemes, optimal error estimates.

1. Introduction

The numerical solution of the wave equation is of fundamental importance to the simulation of time dependent acoustic, electromagnetic, or elastic waves. For such wave phenomena the scalar second order wave equation often serves as a model problem. In the study of wave equations for some physical problems, such as acoustic or elastic waves traveling through heterogeneous media, there can be discontinuities in the coefficients of the equation at interfaces (e.g., [7,8,24] and references therein). For instance, an acoustic wave propagating at different speeds in different media is modeled by the second order wave equation with discontinuous coefficients. This wave propagation is modeled by the linear wave equation

$$u_{tt} - \nabla \cdot \big(\beta(x)\nabla u\big) = f(x,t) \quad \text{in } \Omega \times (0,T] \tag{1.1}$$

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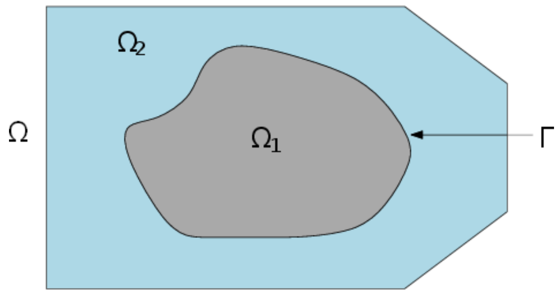

Figure 1: Domain Ω and its sub-domains Ω_1, Ω_2 with interface Γ.

with initial and boundary conditions

$$\begin{aligned} &u(x,0)=u(0),\ \ u_t(x,0)=v(0) && \text{in } \Omega, \\ &u(x,t)=0 && \text{on } \partial\Omega\times(0,T], \end{aligned} \tag{1.2}$$

where $\Omega=\Omega_1\cup\Gamma\cup\Omega_2$ is a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Omega_1\subset\Omega$ is an open domain with Lipschitz boundary $\Gamma=\partial\Omega_1$. Let $\Omega_2=\Omega\backslash\Omega_1$ be an another open domain contained in Ω with boundary $\Gamma\cup\partial\Omega$ (see, Fig. 1). The coefficient function $\beta(x)$ is assumed to be positive and piecewise constant across Γ, i.e., $\beta(x)=\beta_k$ for $x\in\Omega_k$, $k=1,2$. In addition to the usual initial and boundary conditions, u additionally satisfies the following jump conditions on the interface:

$$[u]=\psi, \quad \left[\beta\frac{\partial u}{\partial\eta}\right]=\phi \quad \text{along} \quad \Gamma\times(0,T]. \tag{1.3}$$

Here η is the outward pointing unit normal to Ω_1 and $[v]$ denotes the jump of a quantity v across the interface Γ i.e., $[v](x)=v_1(x)-v_2(x), x\in\Gamma$, where $v_i(x)=v(x)|_{\Omega_i}, i=1,2$. Across the interface Γ, the source function $f:\Omega\times(0,T]\rightarrow\mathbb{R}$ can be singular. We assume that f is sufficiently smooth locally. Jump functions $\psi:\Gamma\times(0,T]\rightarrow\mathbb{R}$ and $\phi:\Gamma\times(0,T]\rightarrow\mathbb{R}$ are given, and $T<\infty$. In this work, it is implicitly assumed that initial data $(u(0),v(0))$ and interface functions (ψ,ϕ) are sufficiently smooth so that solution belongs to desire Sobolev spaces.

In the past few decades there has been remarkable progress in understanding and analyzing numerical algorithms for solving hyperbolic equations. A substantial amount of research on a priori and a posteriori error estimates in the design of finite element methods for the hyperbolic equations without interfaces is available in literature (e.g., [4, 5, 10, 18–22, 26, 30, 32] and references therein). Solving wave propagation problems within heterogeneous media has been of great interest and has drawn significant attention in a variety of fields such as the oil exploration industry and mineral finding as well as the study of earthquakes. Owing to its mathematical complexity and essential importance in a number of application areas, the study of interface problems has evolved into a well defined field in applied and computational mathematics. The past few decades have witnessed intensive research activity in interface problems.