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# Finding Symmetry Groups of Some Quadratic Programming Problems

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**Abstract.** Solution and analysis of mathematical programming problems may be simplified when these problems are symmetric under appropriate linear transformations. In particular, a knowledge of the symmetries may help decrease the problem dimension, reduce the size of the search space by means of linear cuts. While the previous studies of symmetries in the mathematical programming usually dealt with permutations of coordinates of the solutions space, the present paper considers a larger group of invertible linear transformations. We study a special case of the quadratic programming problem, where the objective function and constraints are given by quadratic forms. We formulate conditions, which allow us to transform the original problem to a new system of coordinates, such that the symmetries may be sought only among orthogonal transformations. In particular, these conditions are satisfied if the sum of all matrices of quadratic forms, involved in the constraints, is a positive definite matrix. We describe the structure and some useful properties of the group of symmetries of the problem. Besides that, the methods of detection of such symmetries are outlined for different special cases as well as for the general case.

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## 1. Introduction

Solution and analysis of mathematical programming problems may be simplified when these problems are symmetric under appropriate linear transformations. In particular, a knowledge of the symmetries may help decrease the problem dimension, reduce the size of the search space by means of linear cuts. These methods are applicable in the case of a continuous solutions domain [5, 10, 14] as well as in the integer

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programming [1, 4, 13, 18, 23] and in the mixed integer programming [17, 19]. Other problems in non-convex optimization, complexity theory and sampling that were recently approached by means of continuous symmetries and Lie groups theory may be found in [3, 16]. While most of the applications of symmetries are aimed at speeding up the exact optimization algorithms, yet in some cases the knowledge of symmetries may also be useful in design and analysis of heuristics, in particular, evolutionary algorithms [6, 20].

In the present paper, we study the case of continuous solutions domain. While the previous studies of symmetries in mathematical programming usually dealt with permutations of coordinates of the solutions space [13, 14, 17], the present paper considers a larger group of invertible linear transformations. We study the special case of quadratically constrained quadratic programming problem in $\mathbb{R}^N$: the objective function and the constraints are given by quadratic forms, $A$, and $B_1, \dots, B_M$, respectively, as follows:

$$\begin{cases} x^T A x \to \max, \\ x^T B_1 x \le 1, \\ \dots \\ x^T B_M x \le 1, \end{cases} \tag{1.1}$$

where $x$ is an $N$-component column vector of variables. The superscript $T$ here and below denotes the matrix transposition. In what follows, without loss of generality we assume that $N \times N$ matrices $A, B_i, i = 1, \dots, M$, are symmetric. Note that any matrix can be decomposed into a sum of symmetric matrix $S$ and skew-symmetric matrix $C$, and the quadratic form $x^T C x$ identically equals zero, so this assumption may be easily relaxed.

A more substantial assumption, described in detail in Section 2, allows us to transform the original problem to a new system of coordinates, such that the symmetries may be sought only among orthogonal transformations. In particular, these assumptions are satisfied if the matrix $B_\Sigma := \sum_{i=1}^{M} B_i$ is positive definite. Quadratic programming problems with positive definite matrix $B_\Sigma$ may be found e.g. in radiophysics [7]. The well-known Maximum Cut problem (which is NP-hard) may be reduced to a quadratic programming problem with such property [22] as well.

The results of this paper may also be used for detection of symmetries if some of the problem constraints have the inequality $\le$, some have the inequality $\ge$ and some have the equality sign. We will consider only the inequalities $\le$ for the sake of simplicity.

By a symmetry of problem (1.1) we mean a set of linear transformations

$$x \to y = Px, \tag{1.2}$$

defined by a non-degenerate matrix $P$ such that the problem (1.1), expressed in terms of the transformed space (i.e., through the vector columns $y$), coincides with the original problem. That is, in terms of the vectors $y$ our optimization problem again has the form