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Stability for Constrained Minimax Optimization

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Abstract. Minimax optimization problems are an important class of optimization problems arising from both modern machine learning and from traditional research areas. We focus on the stability of constrained minimax optimization problems based on the notion of local minimax point by Dai and Zhang (2020). Firstly, we extend the classical Jacobian uniqueness conditions of nonlinear programming to the constrained minimax problem and prove that this set of properties is stable with respect to small $\mathcal{C}^2$ perturbation. Secondly, we provide a set of conditions, called Property A, which does not require the strict complementarity condition for the upper level constraints. Finally, we prove that Property A is a sufficient condition for the strong regularity of the Kurash-Kuhn-Tucker (KKT) system at the KKT point, and it is also a sufficient condition for the local Lipschitzian homeomorphism of the Kojima mapping near the KKT point.

AMS subject classifications: 90C30

Key words: Constrained minimax optimization, Jacobian uniqueness conditions, strong regularity, strong sufficient optimality condition, Kojima mapping, local Lipschitzian homeomorphism.

1 Introduction

Let m,n,m_1,m_2,n_1 and n_2 be positive integers and $f:\Re^n\times\Re^m\to\Re$, $h:\Re^n\times\Re^m\to\Re^{m_1}$, $g:\Re^n\times\Re^m\to\Re^{m_2}$, $H:\Re^n\to\Re^{n_1}$ and $G:\Re^n\to\Re^{n_2}$ be given functions. We are interested in the constrained minimax optimization problem of the form

$$\min_{x\in\Phi}\max_{y\in Y(x)} f(x,y), \tag{1.1}$$

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where $\Phi\subset\Re^n$ is a feasible set of decision variable x defined by

$$\Phi=\left\{x\in\Re^n : H(x)=0, G(x)\leq 0\right\} \tag{1.2}$$

and $Y:\Re^n\rightrightarrows\Re^m$ is a set-valued mapping defined by

$$Y(x)=\left\{y\in\Re^m : h(x,y)=0, g(x,y)\leq 0\right\}. \tag{1.3}$$

For unconstrained nonconvex-nonconcave minimax optimization, Jin *et al.* [7] proposed a proper definition of local minimax point. This definition of local minimax point is extended in [5] for the constrained minimax optimization problem (1.1).

Definition 1.1. *A point $(x^*,y^*)\in\Re^n\times\Re^m$ is said to be a local minimax point of problem* (1.1) *if there exist $\delta_0>0$ and a function $\eta:(0,\delta_0]\to\Re_+$ satisfying $\eta(\delta)\to 0$ as $\delta\to 0$ such that for any $\delta\in(0,\delta_0]$ and any $(x,y)\in[\boldsymbol{B}_\delta(x^*)\cap\Phi]\times[Y(x^*)\cap\boldsymbol{B}_{\eta(\delta)}(y^*)]$, we have*

$$f(x^*,y)\leq f(x^*,y^*)\leq\max_z\left\{f(x,z): z\in Y(x)\cap\boldsymbol{B}_{\eta(\delta)}(y^*)\right\}. \tag{1.4}$$

In [5], we established the first-order optimality, the second-order necessary and sufficient optimality conditions for problem (1.1) when the Jacobian uniqueness conditions are satisfied for the lower level problem and the first-order necessary optimality conditions when the strong second-order sufficient optimality condition and the linear independence constraint qualification are satisfied for the lower level problem.

It is well known that, for nonlinear programming, the Jacobian uniqueness condition can be used to establish the stability of the $\mathcal{C}^2$-perturbation (see for instance [6]) and prove that the strong second-order sufficient optimality condition and the linear independence constraint qualification are equivalent to the strong regularity of the Kurash-Kuhn-Tucker (KKT) system (see [8, 11]). The question naturally arises: What are the counterparts of these two stability properties for the constrained minimax optimization problem? The purpose of this paper is to answer this basic question.

The rest of this paper is organized as follows. In Section 2, we develop a simplified version for second-order optimality conditions for the constrained minimax optimization problem, which is suitable for the study of stability properties. In Section 3, we prove that the proposed Jacobian uniqueness conditions for problem (1.1) are kept when a $\mathcal{C}^2$-perturbation of the original problem occurs. In Section 4, we prove that the proposed Property A, which does not require the strict complementarity for the upper level problem, is a sufficient condition for the strong regularity of the KKT system at the KKT point. Finally, we draw a conclusion in Section 5.

Notation. Scalars and vectors are expressed in lower case letters and matrices are expressed in upper case letters. For a vector x, denote $\mathbf{B}_\delta(x)=\{x' : \|x'-x\|\leq\delta\}$. For a, $b\in\Re^p$, $a\circ b$ denotes the Hadamard product of a and b; namely, $a\circ b=(a_1b_1,\ldots,a_pb_p)^T$. For $a\in\Re^p$, $a>0$, denote $\sqrt{a}=\mathrm{Diag}(\sqrt{a_1},\ldots,\sqrt{a_p})$. For a convex set $D\subset\Re^k$, we use $\Pi_D(w)$