Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2022-0236

Vol. **33**, No. 3, pp. 764-794
March 2023

A Fourth-Order Kernel-Free Boundary Integral Method for Interface Problems

Yaning Xie[1], Shuwang Li[2] and Wenjun Ying[3,*]

[1] *College of Science, Zhejiang University of Technology, Xihu, Hangzhou 310023, P.R. China.*
[2] *Department of Applied Mathematics, Illinois Institute of Technology, Chicago, United States.*
[3] *School of Mathematical Sciences, MOE-LSC and Institute of Natural Sciences, Shanghai Jiao Tong University, Minhang, Shanghai 200240, P.R. China.*

Received 18 September 2022; Accepted (in revised version) 1 March 2023

Abstract. This paper presents a fourth-order Cartesian grid based boundary integral method (BIM) for heterogeneous interface problems in two and three dimensional space, where the problem interfaces are irregular and can be explicitly given by parametric curves or implicitly defined by level set functions. The method reformulates the governing equation with interface conditions into boundary integral equations (BIEs) and reinterprets the involved integrals as solutions to some simple interface problems in an extended regular region. Solution of the simple equivalent interface problems for integral evaluation relies on a fourth-order finite difference method with an FFT-based fast elliptic solver. The structure of the coefficient matrix is preserved even with the existence of the interface. In the whole calculation process, analytical expressions of Green's functions are never determined, formulated or computed. This is the novelty of the proposed kernel-free boundary integral (KFBI) method. Numerical experiments in both two and three dimensions are shown to demonstrate the algorithm efficiency and solution accuracy even for problems with a large diffusion coefficient ratio.

AMS subject classifications: 65N06, 65N22, 45B05

Key words: Elliptic interface problem, compact scheme, finite difference method, Cartesian grid method, kernel-free boundary integral method, boundary integral equation.

1 Introduction

This paper concerns about high-order numerical solutions of elliptic interface problems in two and three dimensional space. Typical examples of interface problem arise from

*Corresponding author. *Email addresses:* wying@sjtu.edu.cn (W. Ying), xieyaning@zjut.edu.cn (Y. Xie), sli@math.iit.edu (S. Li)

contact discontinuities for immiscible multiphase fluids with different physical characteristics [1–3] and other multiphase diffusion phenomena [4–7].

For interfaces with complex geometry, body-fitted mesh methods such as those based on conventional finite element methods (FEMs) with standard basis functions [8–10] are considered to be computationally expensive and time-consuming, especially for moving interface problems. As an alternative, embedded methods on Cartesian or adaptive Cartesian grids that do not conform to the interface are preferred to some extent. Clearly, Cartesian grid methods have some advantages over unstructured grid methods like easier grid generation and availability of fast elliptic solvers. Numerical applications on this regard include the extended finite element method (XFEM) with enriched finite element space [11–13], the embedded boundary method with adaptive mesh refinement (AMR) [14, 15], the virtual node method with duplicated bilinear elements [16], the finite difference methods (FDMs) with optimal convergence and many others.

This paper is focused on the Cartesian grid methods based on finite difference discretization. As the solution to the interface problem is discontinuous across the interface, special treatment should be attached at irregular grid nodes to maintain the global numerical accuracy. The immersed boundary method (IBM) describes the interaction between the viscous incompressible fluid with the moving immersed boundaries by a δ-function formulation and results in a first-order or formally second-order smeared solution [17–19]. To preserve the jump conditions in contrast to the numerical smearing, extensive sharp interface methods has been designed for this purpose. The immersed interface method (IIM) gives a second-order finite difference scheme for variable coefficient elliptic PDEs with discontinuous coefficients and singular source. The resulting system by the original IIM is not symmetric positive definite [20, 21]. Some variants have been developed to improve the numerical stability and speed up the algorithm efficiency, such as the finite-element IIM [22], the explicit jump immersed interface method (EJIIM) [23] and the decomposed immersed interface method [24].

Combined with the level set function which serves an indicator for the interface location, the Ghost Fluid Method (GFM) implicitly imposes the interfacial jumps to preserve discontinuities of the solution [25]. The GFM defines both real and ghost values at grid nodes near the interface so that the standard one-phase discretization methods can be naturally applied. The ghost values are regarded as extensions for real values across the interface and constructed by constant or linear extrapolation with the original GFM [1, 25]. In this case, direct discretization based on GFM for the variable coefficient Poisson equation yields a symmetric positive-definite linear system, allowing the use of various fast iterative solvers such as preconditioned conjugate gradient (PCG) method [25, 26]. Derivation for ghost values with extrapolations of higher degrees refers to [2, 27–31], where solution accuracy is improved correspondingly but system symmetry is lost meanwhile.

The novelty of the GFM is of vital importance that jump conditions are incorporated by adding a constant vector to the right hand side of the system, thereby maintaining the coefficient matrix unchanged, compared to the case when the interface does not exist [31].