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ON FINITE ELEMENT APPROXIMATIONS TO A SHAPE GRADIENT FLOW IN SHAPE OPTIMIZATION OF ELLIPTIC PROBLEMS*

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Abstract

Shape gradient flows are widely used in numerical shape optimization algorithms. We investigate the accuracy and effectiveness of approximate shape gradients flows for shape optimization of elliptic problems. We present convergence analysis with a priori error estimates for finite element approximations of shape gradient flows associated with a distributed or boundary expression of Eulerian derivative. Numerical examples are presented to verify theory and show that using the volume expression is effective for shape optimization with Dirichlet and Neumann boundary conditions.

Mathematics subject classification: 65D15, 65N30, 49Q12.
Key words: Shape optimization, Shape gradient, Eulerian derivative, Finite element, Error estimate.

1. Introduction

Shape optimization has wide applications for many fields in computational science, engineering, and industry ([5, 10, 18, 20]). Many numerical methods have been developed to seek "approximate" optimal shapes by virtue of computer simulations. Among them, gradient-type optimization methods are popularly used to iteratively obtain approximately optimal shape designs. For performing a gradient-type algorithm in shape optimization, a so-called shape gradient can be obtained from *Eulerian derivative*, which is derived to measure the rate of change for some objective functional (so-called "shape functional" in shape optimization community) with respect to boundary variations. The process of obtaining an Eulerian derivative is called *shape sensitivity analysis*, a classic mathematical tool in shape optimization [5, 20]. The Eulerian derivatives can have two formulations [5]: distributed type and boundary type, the latter of which obtained by the Hadamard-Zolésio structure theorem has caused much attention due to its attractively concise appearance. The two forms of Eulerian derivatives are equivalent to each other through integration by parts if the boundary of domain satisfies certain smoothness [5]. Most existing research works on numerical shape optimization algorithms approximate

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the Eulerian derivatives using the popular form of boundary integrals even if this type of Eulerian derivative to be approximated actually fail to hold when the boundary is not smooth enough. The distributed Eulerian derivative holds more generally as for regularity requirement of the boundary [5,13] and deserves to be used [2]. At discrete level, moreover, the distributed Eulerian derivative does not coincide with the surface type even if the boundary is smooth enough.

Finite element methods ([3]) are popular approaches for discretizations of state and its adjoint partial differential equation (PDE) constraint for shape optimization (see, e.g., [10,18]). One main reason is that finite element methods can be used to solve PDEs on arbitrary shaped domains and thus are flexible to domain variations in shape optimization. Other approaches such as boundary element methods [9] are also effective. The accuracy of approximate shape gradients should be essential for implementation of numerical optimization algorithms as stated in [5]. After discretization, the convergence analysis for finite element approximations of the two types of shape gradients was an open problem. Recently, the finite element approximation of distributed shape gradient in shape optimization of elliptic problems satisfying Dirichlet boundary condition was shown to converge faster and be more accurate than that of the surface shape gradient [13]. The domain expressions of Eulerian derivatives are advocated for numerical shape optimization algorithms in tomography [12, 14], topology optimization [4], etc. Comparisons on numerical shape gradient algorithms between the two expressions demonstrated in some sense (starting with a same initial shape) the more effectiveness of the volume expression as shown for eigenvalue optimization [22] and shape design in fluids [15] with Dirichlet boundaries being optimized. A priori error estimates were shown for finite element approximations of shape gradients in eigenvalue problem [21] and shape design of Stokes flows [23]. However, the numerical accuracy of the boundary type shape gradient is not worse than that of the volume type on the discrete level. We refer to the numerical results in [13] for Neumann boundary value problems and to [8] for Neumann and Dirichlet (boundary correction used) boundaray value problems.

Shape gradient flows are usually introduced in numerical shape optimization algorithms to increase smoothness of deformation fields as well as to enhance flexibilities of domain variations through moving mesh grids of the domain inside (see, e.g., [6,14,18,19]). Convergence analysis was given on finite element approximations of shape gradient flows in a distributed H^1 space for eigenvalue optimization [24] and optimal shape design in fluids [16]. The analysis therein substantiates in some sense the fact that more effectiveness of the shape gradient flow associated with domain expression of Eulerian derivative for shape optimization [15, 22]. In this paper, we continue to consider elliptic boundary value problems in shape optimization and present convergence analysis for finite element approximations of shape gradient flows. The a priori error estimates presented here for the "regularized" discrete shape gradients in a distributed H^1 space are new and different from those analysis results of approximate shape gradients in the boundary L^2 space [13]. Moreover, we compare numerical performances for a shape gradient flow in shape optimization. Numerical evidence is provided to show the more effectiveness of distributed shape gradient flow.

The rest of the paper is organized as follows. In Section 2 we consider shape optimization model problems constrained by an elliptic boundary value problem. We present finite element discretizations of a shape functional. In Section 3, we present a priori error estimates for approximate shape gradients flows associated with both distributed and boundary expressions of Eulerian derivatives. In Section 4 numerical examples are presented. Brief conclusions follow