East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2022-315.200423

Vol. **13**, No. 3, pp. 717-739
August 2023

Time Complexity Analysis of Quantum Difference Methods for Multiscale Transport Equations

Xiaoyang He[1], Shi Jin[1,2,*] and Yue Yu[1]

[1]*School of Mathematical Sciences, Institute of Natural Sciences, MOE-LSC, Shanghai Jiao Tong University, Shanghai 200240, China.*
[2]*Shanghai Artificial Intelligence Laboratory, Shanghai, China.*

Received 12 November 2022; Accepted (in revised version) 20 April 2023.

Dedicated to Professor Tao Tang on the occasion of his 60th birthday.

Abstract. We investigate time complexities of finite difference methods for solving the multiscale transport equation by quantum algorithms. It is found that the time complexity of classical and quantum treatments of the standard explicit scheme scale is $\mathcal{O}(1/\epsilon)$, where ϵ is a small scaling parameter. On the other hand, the complexity of the even-odd parity based asymptotic-preserving (AP) schemes do not depend on ϵ. This indicates that in quantum computing, AP schemes (and probably other multiscale ones) are of great importance for solving multiscale transport and kinetic equations.

AMS subject classifications: 81Q15

Key words: Quantum difference methods, multiscale transport equations, asymptotic-preserving schemes, time complexity.

1. Introduction

Transport equations arise in many important applications, from medical imaging, astrophysics, nuclear reactor, to wave propagation in random media and semiconductor device modeling [4,5,25,26]. These equations model probability distribution of particles in a background medium, thus are defined in phase space, suffering from curse-of-dimensionality. In addition, the problem may encounter multiple temporal and spatial scales, and the numerical resolution of the small scales will further increase the computational cost tremendously. Despite of rapid development of multiscale methods, high dimensionality and multiple scales could still pose a major challenge for numerical simulations for transport, and more generally, kinetic equations by classical computers.

On the other hand, quantum computers, in various instances, have been shown to exhibit potential polynomial and even exponential advantage over the classical computers,

*Corresponding author. *Email addresses:* hexiaoyang@sjtu.edu.cn (X. He), shijin-m@sjtu.edu.cn (S. Jin), terenceyuyue@sjtu.edu.cn (Y. Yu)

if one designs adequate quantum algorithms. One of such possibilities is linear algebra problems [6,10,12]. After numerical discretizations, ordinary and partial differential equations can also be formulated as linear algebra problems thus can also use quantum linear algebra solvers to gain quantum advantages in dimension, precision, and the evolution time [2,7,9,14,16,19,22,23]. Most of these works aim at producing quantum state, after which a measurement is needed to extract classical data. In [14] though, physical observables are obtained with possible quantum advantage.

In particular, in [16], for a linear hyperbolic relaxation system with possibly stiff relaxation, it shows that a good multiscale scheme – in this case the popular in kinetic community asymptotic-preserving (AP) scheme, has shown its advantage for quantum algorithms over standard non-AP schemes. Specifically, the numerical complexity that depends on the reciprocal of the small physically scaling scales is great relaxed: the complexity of AP quantum algorithms is independent of the small scaling parameter.

In this article we study the multiscale linear transport equation

$$\epsilon\partial_t f + v\partial_x f = \frac{1}{\epsilon}\left(\frac{1}{2}\int_{-1}^{1} f\,dv' - f\right), \quad x_L < x < x_R, \quad -1 \le v \le 1, \tag{1.1}$$

where $f = f(t,x,v)$ is the probability density distribution for particles at space point $x \in \mathbb{R}$, time t, and $v \in (-1,1)$ is the cosine of the angle between the particle velocity and the x-axis. Comparing with the work in [16], here the equation is in the phase space, and one needs to also discretize the velocity (or angle) variable, and to deal with the nonlocal collision operator, hence further complicating the development of numerical approximations and the study of their time complexity for quantum algorithms. Our goal is to compare the time complexity of quantum algorithms based on an AP scheme [17] and a standard (explicit, thus not AP) scheme and show that the former has a complexity independent of the small physical scaling parameter ϵ while the latter depends on it. Hence, it demonstrates that multiscale methods still make a big difference in terms of time complexity even for quantum algorithms.

Since our aim is to compare the difference in dependence of ϵ, in this article we will only study the spatially one dimensional equation. Quantum advantages in spatial dimensions for numerical methods of partial differential equations have been well studied in other literature, see for examples [7,15,16,21].

Compared with the earlier work [16], where a multiscale hyperbolic relaxation system was studied, here in the time complexity analysis for transport equation defined in the phase space with a nonlocal collisional term, the analytic difficulty is to give a lower bound of the minimum singular value of the coefficient matrix. When neglecting the nonlocal term, one easily observes that the problem is reduced to the prototype problem for fixed velocity variable. Its simplicity allows one to estimate the singular values of the coefficient matrix directly, in which the proof ultimately boils down to the upper bound of the 2-norm