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Wavelet Estimation for Regression Convolution Model with Heteroscedastic Errors

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Abstract. This paper considers an unknown functional estimation problem in a multidimensional periodic regression convolution model with heteroscedastic errors. This model has potential applications in signal recovery when both noise and blur are present in the observed data. Our approach is mainly theoretical, however. We first propose a linear wavelet estimator and then discuss the upper bound for its mean integrated squared error over Besov balls. Moreover, the rate of convergence of this estimator under pointwise error is considered. A nonlinear wavelet estimator is constructed by using the hard thresholding method for adaptivity purposes. It should be pointed out that the obtained rate of convergence of the nonlinear estimator is kept the same as the linear one up to a logarithmic term.

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1 Introduction

In this paper, a multidimensional periodic regression convolution model is considered. To begin, let us describe it mathematically. For a certain positive integer n, we observe n^d independent pairs of random variables $\{(X_i,Y_i)\}_{i\in\{1,\ldots,n\}^d}$, where for any $i\in\{1,\ldots,n\}^d$,

$$Y_i=(f\star g)(X_i)+\xi_i. \tag{1.1}$$

It is supposed that

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- f is an unknown square integrable one-periodic function on $[0,1]^d$, g is a known square integrable one-periodic function on $[0,1]^d$, and $(f\star g)(x)$ denotes the convolution product of f and g that can be defined in this one-periodic setting by

$$(f\star g)(x)=\int_{[0,1]^d} f(x_1-u_1,\ldots,x_d-u_d)g(u_1,\ldots,u_d)du_1\ldots du_d,$$

  with $x=(x_1,\ldots,x_d)\in[0,1]^d$ due to the joint one-periodicity of f and g, making $f\star g$ one-periodic too. A typical example in the case $d=1$ is when f (or g) is of the form $f(x)=\sum_{k=0}^{\infty}a_k[\sin(2\pi x)]^{\mu_k}[\cos(2\pi x)]^{\delta_k}$, where $(a_k)_{k\in\mathbb{N}}\in\mathbb{R}$, $(\mu_k)_{k\in\mathbb{N}}\in\mathbb{N}$ and $(\delta_k)_{k\in\mathbb{N}}\in\mathbb{N}$ are possible unknown sequences. Such an expression is standard in the setting of sinusoidal regression, among other regression settings.

- for any $i=(i_1,\ldots,i_d)\in\{1,\ldots,n\}^d$, we have

$$X_i=(X_{i_1}^{(1)},\ldots,X_{i_d}^{(d)}),$$

  where, for any $(u,v)\in\{1,\ldots,d\}^2$, $X_v^{(u)}$ has the uniform distribution on $[0,1]$ and $(\xi_i)_{i\in\{1,\ldots,n\}^d}$ are independent zero mean random variables. All the random variables are independent.

- there exists a known sequences of real numbers $(\sigma_i^2)_{i\in\{1,\cdots,n\}^d}$ such that, for any $i\in\{1,\cdots,n\}^d$,

$$\mathbb{V}(\xi_i)=\mathbb{E}\left(\xi_i^2\right)=\sigma_i^2.$$

  Thus, the heteroscedastic case is considered. The distributions of $(\xi_i)_{i\in\{1,\cdots,n\}^d}$ and possible finite moments of order $\delta>2$ are unknown.

We aim to estimate f from $\{(X_i,Y_i)\}_{i\in\{1,\cdots,n\}^d}$.

The model (1.1) is an inverse problem from instrumental regression. It has several applications in economics, see Hausman and Newey [1], Newey and Powell [2], and Florens [3]. Statistical results can be found in Hall and Horowitz [4], Darolles *et al.* [5], Florens *et al.* [6], Han *et al.* [7] and Jeon *et al.* [8, 9]. For the form (1.1), it was considered in Bissantz and Birke [10] and Birke *et al.* [11] under a deterministic design, and in Hildebrandt *et al.* [12] with a random design. Chesneau and Kachour [13] studied the estimation of the derivatives of a function in one-dimension with random design. It has also potential of application in signal denoising where periodic functions and convolution are standard tools. However, to the best of our knowledge, there is no result on the estimation of f in the multidimensional case and (or) under our weak assumptions on $(\xi_i)_{i\in\{1,\cdots,n\}^d}$ (heteroscedastic case and finite moment of order 2 only). In this case, it has potential applications in signal recovery when both noise and blur are present in the observed data. It is particularly of interest for $d=2$, in an image deblurring setting (astronomical images, medical images, etc.). On this topic, we may refer to the applied works of Schulz [14], Hall and Qiu [15], and Qiu and Kang [16].