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Global Dynamics of the Cahn-Hilliard/Allen-Cahn Equation

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Abstract. In this paper, we consider the global dynamics of the Cahn-Hilliard/Allen-Cahn equation with periodic boundary value conditions in 2D bounded domain Ω. We show that the equation has a global attractor in $H^4_{per}(\Omega)$ when the initial value belongs to $H^1_{per}(\Omega)$.

AMS subject classifications: 35B40, 35Q35, 76W05

Key words: Global attractor, Cahn-Hilliard/Allen-Cahn, absorbing set

1 Introduction

Consider the following Cahn-Hilliard/Allen-Cahn equation

$$u_t = -\Delta[\gamma\Delta u - f(u)] + [\gamma\Delta u - f(u)], \quad \gamma > 0, \ x \in \Omega, \tag{1.1}$$

where $\Omega = (0,L_1)\times(0,L_2), L_1, L_2 > 0$ is a bounded domain in $\mathbb{R}^2$, γ is a positive constant. In Eq. (1.1), $u(x,t)$ denotes the coverage and $f(u)$ is the derivative of $F(u)$, which is a double-well potential with wells ± 1, satisfies $F(u) = \frac{1}{4}(u^2-1)^2$, the Cahn-Hilliard term corresponds to surface diffusion, while the Allen-Cahn to adsorption/desorption. It is worth point out that when the term $\gamma\Delta u - f(u)$ is absent, then (1.1) becomes the well-known Cahn-Hilliard equation

$$u_t - \gamma\Delta^2 u = \Delta f(u),$$

which was studied by many authors (see [1–4, 11, 14] and the reference therein).

On the basis of physical considerations, Eq. (1.1) is supplemented with the periodic boundary value conditions

$$\varphi|_{x_{i=0}} = \varphi|_{x_{i=L_i}}, \quad i = 1,2, \tag{1.2}$$

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for u and the derivatives of u at least of order ≤ 3, and the initial condition

$$u(x,0)=u_0(x), \quad x\in\Omega. \tag{1.3}$$

During the past years, many classical results on the Cahn-Hilliard/Allen-Cahn equation were established. It was Karali and Katsoulakis [7] who first introduced the Cahn-Hilliard/Allen-Cahn equation as a simplified mesoscopic model for pattern formation mechanisms of surface processes. Latter, Israel [5,6] not only studied the well-posedness and long time behavior of solutions to Eq. (1.1) with a singular potential F and dynamic boundary conditions, but also considered the well-posedness, global attractors and exponential attractors for Eq. (1.1) with polynomial potential F and Dirichlet boundary conditions. Moreover, for the Cahn-Hilliard/Allen-Cahn equation with regular potential F and Neumann boundary conditions, Karali et. al. [8,9] considered the existence of solution and the convergence of the equation. The authors structured the sequences of solutions converging to the second order Allen-Cahn equation. Recently, Liu and his collaborators also focus on Eq. (1.1) (see [10,13,15,16]). In [15], Zhang and Liu established the existence of weak solution for Cahn-Hilliard/Allen-Cahn equation with degenerate mobility. Besides, the existence of time periodic of solutions was studied by Liu and Tang [10], the optimal control problem was finished by Zhang, Li and Liu [16]. We remark that Tang, Liu and Zhao [13] studied the existence of global attractor in $H^k(\Omega)$ $(0\leq k<5)$ provided that the initial data $u_0\in H^k(\Omega)$ and Ω is a smooth bounded domain in $\mathbb{R}^n$ $(n\leq 2)$.

There are two previous papers related to the global attractor of Cahn-Hilliard/Allen-Cahn equation. The first one is [6]. The authors proved the existence of global attractor in $H^{-1}(\Omega)$. Latterly, this result was improved by Tang, Liu and Zhao [13]. By using iterative technique and the properties of semigroup, assume that $u_0\in H^k(\Omega)$, the authors proved the existence of global attractor in $H^k(\Omega)$ $(0\leq k<5)$. In this paper, we continue to study the global attractor for Cahn-Hilliard/Allen-Cahn equation. The main purpose of this paper is to obtain the existence of global attractor in $H^4(\Omega)$ provided that the initial data u_0 only in $H^1(\Omega)$. It is easy to see that our result can be seen as an improvement of the previous results in [6,13].

In this paper, we denote by $H=L^2(\Omega)$, $(\cdot,\cdot)$ the H-inner product and by $\|\cdot\|$ the corresponding H-norm, denote $A=-\Delta$, where $-\Delta$ is the Laplace operator. Assume $\int_\Omega u_0(x)dx=0$, then $\int_\Omega u(x,t)dx=0$ for $t>0$. Seeting

$$\dot{H}^k_{per}=\{u|u\in\dot{H}^k_{per}(\Omega),\int_\Omega u(x,t)dx=0\}, \quad k=1,2,\cdots.$$

The following results on the existence of global weak solution can be found in [8,13].

Lemma 1.1. *Suppose that $u_0\in\dot{H}^1_{per}(\Omega)$, the problem* (1.1)-(1.3) *has a global weak solution, such that*

$$u\in L^\infty([0,T];\dot{H}^1_{per}(\Omega))\cap L^2([0,T];\dot{H}^3_{per}(\Omega)).$$

Using Lemma 1.1, we can define the operator semigroup

$$S(t)u_0:\dot{H}^1_{per}(\Omega)\times R^+\to\dot{H}^1_{per}(\Omega)$$