Ann. Appl. Math.
doi: 10.4208/aam.OA-2023-0010

Vol. **39**, No. 2, pp. 181-205
May 2023

Nonnegative Low Rank Matrix Completion by Riemannian Optimalization Methods

Guang-Jing Song[1] and Michael K. Ng[2,*]

[1] *School of Mathematics and Information Sciences, Weifang University, Weifang, Shandong 261061, China*
[2] *Department of Mathematics, The University of Hong Kong, Hong Kong, China*

Received 18 April 2023; Accepted (in revised version) 5 May 2023

Dedicated to the memory of Professor Zhongci Shi

Abstract. In this paper, we study Riemannian optimization methods for the problem of nonnegative matrix completion that is to recover a nonnegative low rank matrix from its partial observed entries. With the underlying matrix incohence conditions, we show that when the number m of observed entries are sampled independently and uniformly without replacement, the inexact Riemannian gradient descent method can recover the underlying n_1-by-n_2 nonnegative matrix of rank r provided that m is of $\mathcal{O}(r^2 s\log^2 s)$, where $s=\max\{n_1,n_2\}$. Numerical examples are given to illustrate that the nonnegativity property would be useful in the matrix recovery. In particular, we demonstrate the number of samples required to recover the underlying low rank matrix with using the nonnegativity property is smaller than that without using the property.

AMS subject classifications: 15A23, 65f22

Key words: Manifolds, tangent spaces, nonnegative matrices, low rank.

---

*Corresponding author.
Emails: `sgjshu@163.com` (G. Song), `mng@maths.hku.hk` (M. Ng)

1 Introduction

Matrix completion, the problem of filling the missing elements by partially observed matrices became popular after the Netflix prize competition which was held in 2006. In order to avoiding being an underdetermined and intractable problem, low rank is often a necessary hypothesis to restrict the degree of freedoms of the missing entries. The matrix completion problem can be formulated as the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & \text{rank}(X) \\ \text{subject to} \quad & P_\Omega(X)=P_\Omega(A), \end{aligned} \tag{1.1}$$

where $X\in\mathbb{R}^{n_1\times n_2}$ is the decision variable, the set Ω of locations corresponding to the observed entries($(i,j)\in\Omega$ if A_{ij} is observed) is a set of cardinality m sampled uniformly at random, and the corresponding sampling operator P_Ω is defined by

$$[P_\Omega(X)]_{i,j}=\begin{cases} X_{ij}, & \text{if} \quad (i,j)\in\Omega, \\ 0, & \text{otherwise.} \end{cases}$$

In general, the rank minimization problem listed in (1.1) is NP-hard and computationally intractable. Many methods were proposed to solve the matrix completion problem, see for instance [1–4, 6–14]. In general, it can be divided into two categories: convex and non-convex optimization methods. Under the framework of convex optimization, the nuclear norm minimization problem

$$\begin{aligned} \text{minimize} \quad & \|X\|_* \\ \text{subject to} \quad & P_\Omega(X)=P_\Omega(A), \end{aligned} \tag{1.2}$$

is often applied to recover the unknown matrix entries, where the nuclear norm $\|X\|_*$ of a matrix X is defined as the sum of its singular values. With some suitable assumptions (incoherence conditions), it has been shown that if the number of observed entries satisfies $m\sim\mathcal{O}(sr^2\log^\alpha s)$ for some $\alpha\geq 0$, the underlying rank r matrix can be exactly recovered with high probability, where $s=\max\{n_1,n_2\}$. Meanwhile, many computationally efficient algorithms are designed to solve model (1.2), see [15–18] and references therein. On the other hand, there are non-convex optimization methods for solving (1.1) by parameterizing in a factorization form or studying in a set of fixed rank matrices. The computational cost of most non-convex algorithms are shown to be cheaper than that of the convex methods. The major issue is how to choose suitable initial guesses in non-convex optimization methods such that they can converge to the underlying low rank solution.