Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.OA-2022-0143

Vol. **16**, No. 3, pp. 792-819
August 2023

The MFE-CFE-GFE Method for the Fully Coupled Quasi-Static Thermo-Poroelastic Problem

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Received 24 August 2022; Accepted (in revised version) 15 March 2023

Abstract. In this work, we consider a combined finite element method for fully coupled nonlinear thermo-poroelastic model problems. The mixed finite element (MFE) method is used for the pressure, the characteristics finite element (CFE) method is used for the temperature, and the Galerkin finite element (GFE) method is used for the elastic displacement. The semi-discrete and fully discrete finite element schemes are established and the stability of this method is presented. We derive error estimates for the pressure, temperature and displacement. Several numerical examples are presented to confirm the accuracy of the method.

AMS subject classifications: 35M10, 65M12, 65M60, 74F05, 74F10

Key words: Quasi-static thermo-poroelasty, characteristics finite element method, porous media, numerical experiments.

1. Introduction

The theory of thermo-poroelasticity mathematically describes the coupled process between the deformation of porous materials and the internal fluid flow under the non-isothermal case. There are three primary variables in the mathematical modeling [7], which are the fluid pressure, the temperature and the elastic displacement of the solid. Due to the wide range of applications in biomedical engineering, petroleum engineering, environment engineering and geothermal survey, the study of thermal-poroelastic model has become more and more popular. In fact, the thermal-poroelastic model is derived from the Biot's model, which can be seen as a coupling process between the Biot's model and the heat equation. In recent years, the theory of Biot's model is well developed. Terzhagi [23] and Biot [1, 2] constitute the theoretical basis of Biot's problem. On this basis, extensive literature has studied numerical methods on the Biot's model, such as the finite difference method [11, 13], the finite element method [3, 16–18], the mixed finite element [19, 20].

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Due to the thermo-poroelastic problem's complex coupled nature, there are only a few literature to study the analysis and numerical simulation of it. By analysing the method of two-scale expansions, the fully coupled thermo-poroelastic model was derived by Brun [6], who later investigated the priori energy estimates and the regularity properties of the solutions [6]. In [15], using the sequential iteration method, Kolesov analyzed the decoupled method for the thermo-poroelastic problem. The sequential iteration method of this problem was presented by Kim [14]. Then Brun [5] studied the coupled nonlinear thermo-poroelasticity model problem using the monolithic and splitting-based iterative procedures, which split the coupled problem into a set of simpler subproblems. For three sub-problems with different strong/weak coupling strength, this article [5] present six different algorithms to compare the applicability of six algorithms to different coupled subproblems. Since the article [5] does not describe in detail the convergence order of six algorithms in different coupling cases, the standard Galerkin method for the thermal-porosity-elasticity model problem is proposed in [27] and the convergence order of this method in different coupling cases is obtained.

The thermo-poroelasticity problem we consider consists of three subproblems, for the mass balance equation, the energy balance equation, and the momentum balance equation. We apply the characteristics finite element (CFE) method for the temperature T of the energy balance equation because the energy balance equation is a convective-diffusion equation with a nonlinear convective transport term. When the effective thermal conductivity $\boldsymbol{\Theta}$ is set as a small value, the temperature equation is that advection dominates diffusion. Since the standard Galerkin finite element [27] method applied to the convection-dominated problems does not obtain pretty results, and produces non-physical oscillation or excessive numerical diffusion, we need to find a more efficient and stable method. The characteristics finite element method was initiated by Douglas and Russell [9] in 1982. This efficient method which solved convection-dominated flows was applied to many problems, including Navier-Stokes equations [21], miscible displacement in porous media problems [8, 10, 12, 22, 24, 26], etc.

In this article, the thermo-poroelasticity problem is a fully coupled problem, with nonlinear terms, therefore we present a mixed element method for the pressure and Darcy velocity of the mass balance equation, a characteristics finite element method is used for the temperature of the energy balance equation, and a Galerkin finite element method for the elastic displacement of the momentum balance equation. By using this combined method, we establish the semi-discrete and full discrete scheme and prove the error estimates for the displacement, pressure and temperature. For the different strengths of the coupling between the different subproblems, some numerical examples are carried out to check the accuracy and efficiency of the method.

The rest of this paper is organized as follows. In Section 2, we introduce the fully coupled thermo-poroelasticity model and some notations. The corresponding semi-discrete scheme and fully discrete scheme of the thermo-poroelasticity model will be shown in Section 3. In Section 4, we show the stability and uniqueness of the solution. The convergence analysis and error estimates for the semi discrete system and fully dis-