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Numerical Solution of the Incompressible Navier-Stokes Equation by a Deep Branching Algorithm

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Abstract. We present an algorithm for the numerical solution of systems of fully nonlinear PDEs using stochastic coded branching trees. This approach covers functional nonlinearities involving gradient terms of arbitrary orders, and it requires only a boundary condition over space at a given terminal time T instead of Dirichlet or Neumann boundary conditions at all times as in standard solvers. Its implementation relies on Monte Carlo estimation, and uses neural networks that perform a meshfree functional estimation on a space-time domain. The algorithm is applied to the numerical solution of the Navier-Stokes equation and is benchmarked to other implementations in the cases of the Taylor-Green vortex and Arnold-Beltrami-Childress flow.

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1 Introduction

This paper is concerned with the numerical solution of systems of $d+1$ fully nonlinear coupled parabolic partial differential equations (PDEs) and Poisson equations on $[0,T]\times\mathbb{R}^d$, of the form

$$\begin{cases} \partial_t u_i(t,x)+\nu\Delta u_i(t,x)+f_i\big(\partial_{\bar{\alpha}^1}u_0(t,x),\cdots,\partial_{\bar{\alpha}^q}u_0(t,x),\partial_{\bar{\alpha}^{q+1}}u_{\beta_{q+1}}(t,x),\cdots,\partial_{\bar{\alpha}^n}u_{\beta_n}(t,x)\big)=0,\\ \Delta u_0(t,x)=f_0\big(\partial_{\bar{\alpha}^{q+1}}u_{\beta_{q+1}}(t,x),\cdots,\partial_{\bar{\alpha}^n}u_{\beta_n}(t,x)\big),\\ u_i(T,x)=\phi_i(x),\quad (t,x)=(t,x_1,\cdots,x_d)\in[0,T]\times\mathbb{R}^d,\quad i=1,\cdots,d, \end{cases} \tag{1.1}$$

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where $q \in \{0,1,\cdots,n\}$, $\partial_t u(t,x) = \partial u(t,x)/\partial t$, $\nu > 0$, $\Delta = \sum_{i=1}^d \partial^2/\partial x_i^2$ is the standard d-dimensional Laplacian, $1 \leq \beta_j \leq d$ for $q < j \leq n$, $\bar{\alpha}^i = (\alpha_1^i,\cdots,\alpha_d^i) \in \mathbb{N}^d$, $i=0,1,\cdots,n$, $f_i(x_1,\cdots,x_n)$ and $f_0(x_{q+1},\cdots,x_n)$ are smooth functions of the derivatives

$$\partial_{\bar{\alpha}^i} u(t,x) = \frac{\partial^{\alpha_1^i}}{\partial x_1^{\alpha_1^i}} \cdots \frac{\partial^{\alpha_d^i}}{\partial x_d^{\alpha_d^i}} u(t,x_1,\cdots,x_d), \quad (x_1,\cdots,x_d) \in \mathbb{R}^d,$$

and $(\phi_i)_{i=1,\cdots,d}$ is a smooth terminal condition. We note that the problem (1.1) is posed using the terminal time boundary condition $u_i(T,x) = \phi_i(x)$ in $(x_1,\cdots,x_d) \in \mathbb{R}^d$, instead of assuming Dirichlet and Neumann boundary conditions at all times as is usually the case in the finite difference and mesh-based literature.

As is well known, standard numerical schemes for solving (1.1) by e.g. finite differences or finite elements suffer from a high computational cost which typically grows exponentially with the dimension d. This motivates the study of probabilistic representations of (1.1), which, combined with meshfree Monte Carlo approximation, can overcome the curse of dimensionality. In addition, it is not clear how the standard numerical schemes can be applied when boundary conditions are not available.

Probabilistic representations for the solutions of first and second order nonlinear PDEs can be obtained by writing $u(t,x) \in \mathbb{R}$ as $u(t,x) = Y_t^{t,x}$, where $(Y_s^{t,x})_{t \leq s \leq T}$ is the solution of first or second order backward stochastic differential equation (BSDE), see [9, 13, 30, 35] for a deep learning implementation. See also [23] for the use of stochastic branching processes for the probabilistic representation of solutions of the Navier-Stokes equation.

Stochastic branching diffusion mechanisms [19, 27, 33] have also been applied to the probabilistic representation of the solutions of nonlinear PDEs, see e.g. [14, 15] for the case of polynomial first order gradient nonlinearities, and [11, 12, 16, 36] for finite difference schemes combined with Monte Carlo estimation for fully nonlinear PDEs with gradients of order up to two. However, extending the above approaches to nonlinearities in higher order derivatives involves technical difficulties linked to the integrability of the Malliavin-type weights used in repeated integration by parts argument, see page 199 of [15].

In [29], a stochastic branching method that carries information on (possibly functional) nonlinearities along a random tree has been introduced, with the aim of providing Monte Carlo schemes for the numerical solution of fully nonlinear PDEs with gradients of arbitrary orders on the real line. This method has been implemented on $\mathbb{R}^d$ in [28] using a neural network approach to efficiently approximate the PDE solution $u(t,x) \in \mathbb{R}$ over a bounded domain in $[0,T] \times \mathbb{R}^d$.

In this paper, we extend the approaches in [28, 29] to treat the case of systems of fully nonlinear PDEs of the form (1.1), and we apply our algorithm to the incompressible