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Nonisospectral Lotka–Volterra Systems as a Candidate Model for Food Chain

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Dedicated to the memory of Professor Zhongci Shi

Abstract. In this paper, we derive a generalized nonisospectral semi-infinite Lotka–Volterra equation, which possesses a determinant solution. We also give its a Lax pair expressed in terms of symmetric orthogonal polynomials. In addition, if the simplified case of the moment evolution relation is considered, that is, without the convolution term, we also give a generalized nonisospectral finite Lotka–Volterra equation with an explicit determinant solution. Finally, an application of the generalized nonisospectral continuous-time Lotka–Volterra equation in the food chain is investigated by numerical simulation. Our approach is mainly based on Hirota's bilinear method and determinant techniques.

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Key words: Nonisospectral Lotka–Volterra, symmetric orthogonal polynomials, food chains, determinant techniques, Hirota's bilinear method.

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1 Introduction

It is known that Lotka–Volterra equation is an important biological model. Food chains with n members modelled by Lotka–Volterra equation are as follows [25]

$$\begin{cases} \dot{x}_1 = x_1(r_1 - a_{11}x_1 - a_{12}x_2), \\ \dot{x}_j = x_j(-r_j + a_{j,j-1}x_{j-1} - a_{jj}x_j - a_{j,j+1}x_{j+1}), \quad j = 2,3,\cdots,n-1, \\ \dot{x}_n = x_n(-r_n + a_{n,n-1}x_{n-1} - a_{nn}x_n), \end{cases} \tag{1.1}$$

with all $r_j, a_{ij} \geq 0$, where the first population is the prey for the second, which is the prey for the third, and so on up to the n-th, which is at the top of the food pyramid. The x_i denotes the density, r_i is intrinsic growth (or decay) rate, and a_{ij} describes the effect of the j-th upon the i-th population. $a_{jj} > 0$ means species j's population has a negative effect on itself, which attempt to prevent populations from growing indefinitely.

It is noted that in most literatures, the coefficients a_{ij} of (1.1) are usually regarded as constants (see [13]), namely the system (1.1) is often considered as an autonomous system. This assumption of a constant environment is reasonable in some cases. Nevertheless, since biological and environmental parameters are naturally affected by time fluctuations in reality, a more realistic model would allow these parameters to change over time. Thus the studies on the non-autonomous Lotka–Volterra equation appear naturally. For instance, in [37], Teng and Yu considered a two-dimensional non-autonomous prey-predator Lotka–Volterra system, which corresponds to the case $n=2$ of (1.1) with the coefficients r_i and a_{ij} (i,j=1,2) being continuous and bounded functions of t defined on $[0,\infty)$. The extinction of the general non-autonomous predator-prey Lotka–Volterra system under some very weak assumptions is studied there. Besides, Meng and Chen in [31] studied a non-autonomous Lotka–Volterra almost periodic predator-prey dispersal system with discrete and continuous time delaying, which consists of n-patches. This system is uniformly persistent and globally asymptotically stable under some appropriate conditions, and owns a unique globally asymptotical stable strictly positive almost periodic solution. More relevant and rather incomplete examples modeled by the non-autonomous Lotka–Volterra system please consult [1, 12, 20–22, 28, 38].

In this paper, we study the Lotka–Volterra model from the view of integrable system. Apart from an important biological model, the Lotka–Volterra equation is also a classical integrable system. We obtain a nonisospectral Lotka–Volterra equation, which can be regarded as a non-autonomous food chain model with n species. Some qualitative analyses about the impacts of the corresponding coefficients have been done with the help of numerical simulations.