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A POSTERIORI ERROR ANALYSIS FOR AN ULTRA-WEAK DISCONTINUOUS GALERKIN APPROXIMATIONS OF NONLINEAR SECOND-ORDER TWO-POINT BOUNDARY-VALUE PROBLEMS

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This paper is dedicated to the memory of my mother, Rehah Jetlaoui, who died unexpectedly in July 31, 2021 during the completion of the first part of this work [7].

Abstract. In this paper, we present and analyze *a posteriori* error estimates in the L^2-norm of an ultra-weak discontinuous Galerkin (UWDG) method for nonlinear second-order boundary-value problems for ordinary differential equations of the form $u'' = f(x, u)$. We first use the superconvergence results proved in the first part of this paper (*J. Appl. Math. Comput.* 69, 1507-1539, 2023) to prove that the UWDG solution converges, in the L^2-norm, towards a special p-degree interpolating polynomial, when piecewise polynomials of degree at most $p \geq 2$ are used. The order of convergence is proved to be $p + 2$. We then show that the UWDG error on each element can be divided into two parts. The dominant part is proportional to a special $(p+1)$-degree Baccouch polynomial, which can be written as a linear combination of Legendre polynomials of degrees $p - 1$, p, and $p + 1$. The second part converges to zero with order $p + 2$ in the L^2-norm. These results allow us to construct *a posteriori* UWDG error estimates. The proposed error estimates are computationally simple and are obtained by solving a local problem with no boundary conditions on each element. Furthermore, we prove that, for smooth solutions, these *a posteriori* error estimates converge to the exact errors in the L^2-norm under mesh refinement. The order of convergence is proved to be $p + 2$. Finally, we prove that the global effectivity index converges to unity at $\mathcal{O}(h)$ rate. Numerical results are presented exhibiting the reliability and the efficiency of the proposed error estimator.

Key words. Second-order boundary-value problems, ultra-weak discontinuous Galerkin method, superconvergence, *a posteriori* error estimation, Baccouch polynomials.

1. Introduction

This paper is a continuation of our recent paper [7] and is devoted to the *a posteriori* error estimation for the following nonlinear second-order two-point boundary-value problems (BVPs) [4, 9, 15, 18] solved with the ultra-weak discontinuous Galerkin (UWDG) method

$$u'' = f(x, u), \quad x \in \Omega = [a, b], \tag{1a}$$

$$u(a) = u_a, \quad u'(b) = u_b, \tag{1b}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a given smooth function. Precise conditions on f are specified later. We would like to point out that, in the present work, we restrict ourselves to either the mixed Dirichlet-Neumann boundary conditions ($u(a) = u_a$, $u'(b) = u_b$) or periodic boundary conditions ($u(a) = u(b)$, $u'(a) = u'(b)$) to simplify the presentation. The main novelty of this paper is to construct *a posteriori* error estimates of the UWDG method proposed in the first part and to prove the convergence of the proposed error estimators in the L^2-norm.

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Various numerical methods have been proposed to solve nonlinear differential equations in literature, such as the finite difference methods, the time-splitting pseudo-spectral methods, the finite element methods, and the discontinuous Galerkin (DG) methods, to name a few. The main reason for us to establish DG methods is because of its flexibility in handling geometry, exhibiting superconvergence properties, accommodating hp-adaptivity, and high parallel efficiency. The DG method was first proposed by Reed and Hill in 1973 [19] for approximating the scalar neutron equation. This type of finite element method uses a piecewise polynomial basis for both the numerical and test function, and it was originally designed to deal with the first spatial derivative only (see, *e.g.*, [11, 12, 14, 19] for detailed discussions). The original DG method has been developed in several directions over the past few decades. For instance, Cockburn and Shu [13] proposed the so-called local discontinuous Galerkin (LDG) method to solve a wide class of nonlinear convection-diffusion equations with high-order spatial derivatives. By introducing auxiliary variables that reduce the original problem into a lower-order system, typically with first-order spatial derivatives, the LDG methods ensure the stability of the scheme by suitable numerical fluxes embedded with the resulting system. See [17, 25, 26] and references therein for recent developments of the LDG method.

Another streamline of development is motivated by the urge to solve high-order problems, and this includes the ultra-weak discontinuous Galerkin method (UWDG) introduced by Despres [16] for linear elliptic PDEs. The idea of the UWDG method for higher-order equations is to shift all the spatial derivatives through integration by parts to the test function in the weak formulation, and the stability of the scheme is guaranteed by certain numerical fluxes and additional internal penalty terms when necessary.

In recent years, the study of superconvergence and *a posteriori error* estimates of DG methods has been an active research field in numerical analysis, see the monographs by Verfürth [23], Wahlbin [24], and Babuška and Strouboulis [5]. A knowledge of superconvergence properties can be used to (i) construct simple and asymptotically exact *a posteriori* estimates of discretization errors and (ii) help detect discontinuities to find elements needing limiting, stabilization and/or refinement. *A posteriori* error estimates play an essential role in assessing the reliability of numerical solutions and in developing efficient adaptive algorithms. Typically, *a posteriori* error estimators employ the known numerical solution to derive estimates of the actual solution errors. They are also used to steer adaptive schemes where either the mesh is locally refined (h-refinement) or the polynomial degree is raised (p-refinement). For an introduction to the subject of *a posteriori* error estimation see the monograph of Ainsworth and Oden [3].

In [7], we presented and analyzed a superconvergent UWDG method for the model problem (1). We first used a suitable choice of the numerical fluxes to derive optimal L^2-error estimates of the scheme. The order of convergence is proved to be $p+1$ in the L^2-norm, when piecewise polynomials of degree $p \geq 2$ are used. Moreover, we proved that the UWDG solution is superconvergent with order $p+2$ for $p = 2$ and $p+3$ for $p \geq 3$ towards a special projection of the exact solution. Finally, we proved that the UWDG solution and its derivative are superconvergent at the nodes with an order of $\mathcal{O}(h^{2p})$. Our proofs are valid for arbitrary regular meshes using piecewise polynomials with degree $p \geq 2$. Numerical experiments were presented to confirm the sharpness of all the theoretical findings. In this work, we use the results in the first part to construct efficient and reliable *a posteriori* error estimates for the UWDG method. We further prove that the proposed *a*