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Spectral Galerkin Approximation of Fractional Optimal Control Problems with Fractional Laplacian

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Abstract. In this paper spectral Galerkin approximation of optimal control problem governed by fractional elliptic equation is investigated. To deal with the nonlocality of fractional Laplacian operator the Caffarelli-Silvestre extension is utilized. The first order optimality condition of the extended optimal control problem is derived. A spectral Galerkin discrete scheme for the extended problem based on weighted Laguerre polynomials is developed. A priori error estimates for the spectral Galerkin discrete scheme is proved. Numerical experiments are presented to show the effectiveness of our methods and to verify the theoretical findings.

AMS subject classifications: 35Q93, 49M25, 49M41

Key words: Fractional Laplacian, optimal control problem, Caffarelli-Silvestre extension, weighted Laguerre polynomials.

1 Introduction

The goal of this paper is to investigate spectral Galerkin approximation of optimal control problem governed by fractional elliptic equation with fractional Laplacian operator defined by spectral expansion. Let Ω be an open, bounded and connected domain in

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$\mathbb{R}^n$, with Lipschitz boundary $\partial\Omega$. We consider the following fractional optimal control problem:

$$\min_{z\in Z_{ad}} J(u,z) := \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \frac{\mu}{2}\|z\|^2_{L^2(\Omega)} \tag{1.1}$$

subject to

$$\begin{cases} (-\Delta)^s u(x) = f + z, & x\in\Omega, \\ u(x) = 0, & x\in\partial\Omega. \end{cases} \tag{1.2}$$

Here the constraint set of control variable z is defined by

$$Z_{ad} = \left\{ z\in L^2(\Omega) : \int_\Omega z(x)dx \geq 0 \right\}.$$

$\mu > 0$ is the regularization parameter, and u_d is the desired state. The operator $(-\Delta)^s$, $(s\in(0,1))$ is the fractional power of Laplacian operator, which will be defined later.

In recent years, optimal control problem [15,28,32,34] has developed into a hot subject across computational mathematics, applied mathematics and systems science. It has a very wide range of applications in engineering control, medical imaging, aerospace and many other fields. The solution of the optimal control problem is to find a way to achieve the optimal performance index of the control system under the constraint conditions. In various fields of human activities, many problems can be described by the optimal control problem with a partial differential equation as the state equation.

Compared with integer order equations, fractional order differential equations can more accurately describe materials and physical processes with memory and heredity, such as viscoelastic materials, diffusion and heat conduction in porous media, etc. Therefore, more and more scholars pay attention to the discussion and analysis of fractional order problems [16, 23, 25–27, 29, 33]. Although optimal control theory has been developed for many years, fractional optimal control theory is a new field in mathematics. In recent years, many numerical methods and algorithms have been developed to solve fractional order optimal control problems. In [30], Ye and Xu proposed a space-time spectral method to solve the time fractional optimal control problems. In [31], they used the space-time spectral method to solve the optimal control problem of time fractional diffusion equation with integral constraints on state variable. In [15], Li and Zhou use spectral collocation method to solve the optimal control problem of space fractional diffusion equation. In [28], Yang, Zhang, Liu, et al proposed the Jacobi spectral collocation method to solve the time fractional optimal control problem. In [24, 32], the authors discussed the spectral Galerkin approximation of optimal control problem governed by fractional differential equation with control integral constraint. Unlike aforementioned works the weighted Jacobi polynomials are used to approximate the state equation. In finite element method aspects the authors discussed finite element approximation [35] of