Journal of Computational Mathematics
Vol.41, No.6, 2023, 1064–1092.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2112-m2020-0283

HIGH ORDER FINITE DIFFERENCE HERMITE WENO FAST SWEEPING METHODS FOR STATIC HAMILTON-JACOBI EQUATIONS*

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Abstract

In this paper, we propose a novel Hermite weighted essentially non-oscillatory (HWENO) fast sweeping method to solve the static Hamilton-Jacobi equations efficiently. During the HWENO reconstruction procedure, the proposed method is built upon a new finite difference fifth order HWENO scheme involving one big stencil and two small stencils. However, one major novelty and difference from the traditional HWENO framework lies in the fact that, we do not need to introduce and solve any additional equations to update the derivatives of the unknown function ϕ. Instead, we use the current ϕ and the old spatial derivative of ϕ to update them. The traditional HWENO fast sweeping method is also introduced in this paper for comparison, where additional equations governing the spatial derivatives of ϕ are introduced. The novel HWENO fast sweeping methods are shown to yield great savings in computational time, which improves the computational efficiency of the traditional HWENO scheme. In addition, a hybrid strategy is also introduced to further reduce computational costs. Extensive numerical experiments are provided to validate the accuracy and efficiency of the proposed approaches.

Mathematics subject classification: 65M60, 35L65.
Key words: Finite difference, Hermite methods, Weighted essentially non-oscillatory method, Fast sweeping method, Static Hamilton-Jacobi equations, Eikonal equation.

1. Introduction

In this paper, we design and validate high order accurate and efficient Hermite weighted essentially non-oscillatory (HWENO) fast sweeping methods for solving the static Hamilton-Jacobi (HJ) equation

$$\begin{cases} H(\nabla\phi,\mathbf{x})=0, & \mathbf{x}\in\Omega\setminus\Gamma,\\ \phi(\mathbf{x})=g(\mathbf{x}), & \mathbf{x}\in\Gamma\subset\Omega, \end{cases} \tag{1.1}$$

* Received October 26, 2020 / Revised version received October 29, 2021 / Accepted December 22, 2021 / Published online June 23, 2022 /

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where Ω is the computational domain in $\mathbb{R}^d$, $\phi(\mathbf{x})$ is the unknown function in Ω, the Hamiltonian H is a nonlinear Lipschitz continuous function depending on $\nabla\phi$ and $\mathbf{x}$, and the boundary condition is given by $\phi(\mathbf{x}) = g(\mathbf{x})$ on the subset $\Gamma \subset \Omega$. One important example to be considered is the Eikonal equation, taking the form of

$$\begin{cases} |\nabla\phi| = f(\mathbf{x}), & \mathbf{x} \in \Omega \setminus \Gamma, \\ \phi(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \Gamma \subset \Omega, \end{cases} \tag{1.2}$$

where $f(\mathbf{x}) > 0$.

The HJ equations have extensive applications in many different fields, for instance in optimal control, computer vision, differential game and geometric optics, image processing and so on [5,28]. It is well-known that global C^1 solution does not exist for time-dependent HJ equations in the generic situation, even if the initial condition is amply smooth. Singularities in the form of discontinuities would appear in the derivatives of the unknown function, hence it is necessary to define a "weak solution" for the HJ equations. The viscosity solutions of the HJ equations were first introduced by Crandall and Lions in [2].

One popular way to numerically solve the static HJ equations is to treat the problem as a stationary boundary value problem, such as the fast marching method (FMM) [3,21,26] and the fast sweeping method (FSM) [7,8,15,25,33,34] can be applied. Compared with FMM, FSM can be constructed to be high order accurate, and becomes a class of popular and effective methods for solving static HJ equations nowadays. The FSM was first introduced in [1] by Boué and Dupuis, to solve a deterministic control problem with quadratic running cost using Markov chain approximation. Later, Zhao [33] applied the FSM to solve the Eikonal equations. Since then, many high order FSM have been developed to solve static HJ equations. In the framework of finite difference methods, Zhang *et al.* [32] combined the third order finite difference WENO-JP scheme [6] with FSM, and Xiong *et al.* [29] studied fifth order WENO-JP FSM scheme. High order accurate boundary treatments (i.e., Richardson extrapolation and Lax-Wendroff type procedure), which are consistent with high order FSM, have been developed for the inflow boundary conditions in [4,29]. In [16], a competent stopping criterion was recommended for high order FSM. In addition, high order FSM was also investigated in the framework of discontinuous Galerkin (DG) finite element method to solve Eikonal equation, and their numerical performance was shown to be effective and robust [10,12,27,31].

In additional to finite difference WENO and DG methods, high order HWENO methods [11,17–19] have recently gained many attention in solving hyperbolic conservation laws. Both the classical WENO and HWENO methods can achieve the high order accuracy and preserve the essentially non-oscillatory property. The main difference lies in the fact that the HWENO scheme uses the Hermite reconstruction, that involves both the unknown variable ϕ and its first order spatial derivative or first moment in the reconstruction. As a result, the reconstruction stencil becomes more compact, although more storage and some additional work are needed to evaluate the spatial derivatives. The HWENO scheme was first proposed during the construction of a suitable limiter for the DG method [17,18], since it is more compact than the standard WENO scheme. In [19], the HWENO scheme was first used to solve the time-dependent HJ equation, and achieved very good numerical results. Compared with the WENO scheme, its boundary treatment is simpler and the numerical error is smaller with the same mesh, as shown in [19]. The HWENO scheme was later extended to solve the hyperbolic conservation law in the finite difference framework [11], where the same advantages can be observed. Since then, a series of HWENO schemes [24,30,35,36] have been investigated to solve hyperbolic