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A Riemannian Inertial Mann Algorithm for Nonexpansive Mappings on Hadamard Manifolds

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Abstract. The problem of finding fixed points of nonexpansive mappings on Hadamard manifolds is considered in this paper. To solve this kind of problem, we propose a modified Riemannian Mann algorithm with inertial effect. Under the assumption of existence of fixed points of the nonexpansive mapping, the global convergence of the proposed algorithm is established. To show the efficiency of the proposed algorithm, numerical comparisons with some existing algorithms are reported.

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Key words: Nonexpansive mapping, Hadamard manifold, fixed point, inertial Mann algorithm.

1. Introduction

In the past few decades, various different optimization problems on Riemannian manifolds have been investigated by many researchers [1, 4, 7–9, 12, 13, 15, 16, 19, 23–26, 28, 30]. Specially, a complete and simply connected Riemannian manifold of non-positive curvature is called a Hadamard manifold. Let $(\mathcal{H}, g)$ be a Hadamard manifold and dist denote the distance induced by g. A mapping $G : \mathcal{H} \to \mathcal{H}$ is called nonexpansive if it satisfies

$$\mathrm{dist}\big(G(p), G(q)\big) \leq \mathrm{dist}(p, q), \quad \forall p, q \in \mathcal{H}.$$

Let $G : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping. In this paper, we are concerned with the

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problem of finding a fixed point of G, i.e.,

$$G(X) = X, \quad X \in \mathcal{H}. \tag{1.1}$$

In the past few years, some different algorithms have been proposed for approximating fixed points of nonexpansive mappings on Hadamard manifolds. Li *et al.* [14] generalized the Halpern algorithm and the Mann algorithm to Hadamard manifolds. The Riemannian Mann algorithm has the following form:

$$X_{n+1} := \exp_{X_n}\big((1-\alpha_n)\exp_{X_n}^{-1} G(X_n)\big), \quad n \geq 0, \tag{1.2}$$

where $\exp_{X_n} : T_{X_n}\mathcal{H} \to \mathcal{H}$ denotes the exponential mapping on $\mathcal{H}$ and $\exp_{X_n}^{-1} : \mathcal{H} \to T_{X_n}\mathcal{H}$ denotes its inverse mapping. Let $\mathrm{Fix}(G) := \{X \in \mathcal{H} \mid G(X) = X\}$ denote the set of all the fixed points of G. The global convergence of this algorithm was proved under the following assumptions [14, Theorem 4.3]:

$$\mathrm{Fix}(G) \neq \emptyset, \quad 0 < \alpha_n < 1, \quad \sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty.$$

Given a point $u \in \mathcal{H}$, the Riemannian Halpern algorithm has the following form:

$$X_{n+1} := \exp_{u}\big((1-\alpha_n)\exp_{u}^{-1} G(X_n)\big), \quad n \geq 0. \tag{1.3}$$

Let $\Pi_{\mathrm{Fix}(G)}$ denote the metric projection on $\mathrm{Fix}(G)$. The iterative sequence $\{X_n\}$ generated by this algorithm, converges to $\Pi_{\mathrm{Fix}(G)}u$, which is the point in $\mathrm{Fix}(G)$ closest to u, under the following assumptions [14, Theorem 3.6]:

$$\begin{aligned} &\mathrm{Fix}(G) \neq \emptyset, \quad 0 < \alpha_n < 1, \quad \lim_{n\to\infty} \alpha_n = 0, \\ &\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \left(\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \quad \text{or} \quad \lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n} = 0\right). \end{aligned}$$

Chugh *et al.* [3] generalized the Ishikawa algorithm to Hadamard manifolds. The Riemannian Ishikawa algorithm has the following form:

$$\begin{aligned} X_{n+1} &:= \exp_{X_n}\big((1-\alpha_n)\exp_{X_n}^{-1} G(Y_n)\big), \\ Y_n &:= \exp_{X_n}\big((1-\beta_n)\exp_{X_n}^{-1} G(X_n)\big), \end{aligned} \quad n \geq 0. \tag{1.4}$$

This is a two-step iterative procedure, the global convergence of this method was proved under the following assumptions [3, Theorem 3.1]:

$$\begin{aligned} &\mathrm{Fix}(G) \neq \emptyset, \quad 0 < \alpha_n < 1, \quad 0 < \beta_n < 1, \\ &\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty, \quad \sum_{n=0}^{\infty} \beta_n(1-\alpha_n) < \infty, \quad \lim_{n\to\infty} \beta_n < 1. \end{aligned}$$