Journal of Computational Mathematics
Vol.42, No.1, 2024, 1–23.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2202-m2021-0290

ANALYSIS OF THE IMPLICIT-EXPLICIT ULTRA-WEAK DISCONTINUOUS GALERKIN METHOD FOR CONVECTION-DIFFUSION PROBLEMS*

Haijin Wang and Anping Xu
School of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, China
Email: hjwang@njupt.edu.cn, 1020082215@njupt.edu.cn
Qi Tao[1)]
Beijing Computational Science Research Center, Beijing 100193, China
Email: taoqi@csrc.ac.cn

Abstract

In this paper, we first present the optimal error estimates of the semi-discrete ultra-weak discontinuous Galerkin method for solving one-dimensional linear convection-diffusion equations. Then, coupling with a kind of Runge-Kutta type implicit-explicit time discretization which treats the convection term explicitly and the diffusion term implicitly, we analyze the stability and error estimates of the corresponding fully discrete schemes. The fully discrete schemes are proved to be stable if the time-step $\tau \leq \tau_0$, where τ_0 is a constant independent of the mesh-size h. Furthermore, by the aid of a special projection and a careful estimate for the convection term, the optimal error estimate is also obtained for the third order fully discrete scheme. Numerical experiments are displayed to verify the theoretical results.

Mathematics subject classification: 65M12, 65M15, 65M60.
Key words: The ultra-weak discontinuous Galerkin method, Convection-diffusion, Implicit-explicit time discretization, Stability, Error estimate.

1. Introduction

Among the time discretization methods for solving convection-diffusion problems, explicit time discretization results in severe time step restriction, while pure implicit time discretization always requires solving large non-linear systems of equations. In [19], a kind of Runge-Kutta (RK) type implicit-explicit (IMEX) time discretization [1] coupled with the local discontinuous Galerkin (LDG) spatial discretization [7] was studied for one dimensional linear convection-diffusion equations. The corresponding fully discrete IMEX-LDG schemes were proved to be unconditionally stable under the time step restriction $\tau \leq \tau_0$, where τ_0 depends only on the coefficients of convection and diffusion and not on the mesh size. The similar results were also extended to non-linear problems in [20] and to multi-dimensional cases in [21]. Later, the stability of IMEX time discretization combined with the embedded DG method [10], the

* Received October 14, 2021 / Revised version received January 4, 2022 / Accepted February 17, 2022 / Published online December 7, 2022 /
1) Corresponding author

(σ, μ)-family DG method [12] and the directed DG method [14] was investigated in [8, 16, 22], respectively.

The stability mechanism of the aforementioned fully discrete methods lies in that, the anti-dissipation of the explicit discretization for the convection term can be controlled by the stability provided by the implicit discretization for the diffusion term. In this paper, we concern about whether the same mechanism is inherent to the ultra-weak discontinuous Galerkin (UWDG) method coupled with IMEX time discretization.

The UWDG method was developed to solve time dependent partial differential equations (PDEs) containing high order spatial derivatives by Cheng and Shu [5]. Unlike the LDG method, the UWDG method does not introduce any auxiliary variables. The main idea of the UWDG method is to apply integration by parts repeatedly and to move all the spatial derivatives from the trial function to the test function in the weak formulations. The UWDG method has been successfully applied to kinds of high order PDEs. In [9], Fu and Shu designed an energy-conserving UWDG method for the generalized KdV equation. The UWDG method for generalized stochastic KdV equations was studied in [13], for Schrödinger equations was studied in [3,4]. Recently, the UWDG method combined with the LDG method to solve the PDEs with high order spatial derivatives was developed by Tao *et al.* [15,18]. It is worth pointing out that, most of the above works focus on the theoretical analysis for the semi-discrete UWDG method. Although IMEX time discretization is used in numerical experiments, such as in [3, 13], there is no theoretical analysis for the fully discrete IMEX-UWDG scheme. Meanwhile, the error estimates of the semi-discrete UWDG method for convection-diffusion problems [5] are not optimal, but numerical experiments show optimal accuracy. As far as the authors know, there is no theoretical analysis to fill this gap so far.

In this work, we will first present the optimal error estimates of the semi-discrete UWDG scheme for solving one-dimensional linear convection-diffusion equations with periodic boundary conditions. The main technique is a special projection to be defined following from [3]. The projection can eliminate the projection errors involved in the diffusion part, but the projection errors involved in the convection part can not be eliminated, so traditional treatment will lose accuracy. By the aid of the stability provided by diffusion discretization, we obtain optimal error estimates for the semi-discrete UWDG scheme.

We will also perform the analysis of stability and error estimates for some fully discrete IMEX-UWDG schemes. Typically, three specific RK type IMEX schemes coupled with the UWDG spatial discretization will be considered. By energy analysis, we prove similar stability results to that for the IMEX-LDG method in [19]. Different from the LDG method, where the discretization of the diffusion part can be converted into some inner products of auxiliary variables, there are no auxiliary variables which can be used in the UWDG method. With respect to the UWDG discretization, we make full use of the symmetric and dissipative properties, which will help us to build up negative definite quadratic forms about the implicit discretization of diffusion part, so as to obtain the desired stability results. Along the similar line of stability analysis and by the aid of the special projection mentioned above, we also carry out the optimal error estimates for the third order IMEX-UWDG scheme.

The paper is organized as follows. We first present the semi-discrete UWDG scheme for the model problem and give its optimal error estimates in Section 2. Then we give the stability analysis of three specific fully discrete IMEX-UWDG schemes in Section 3. In Section 4, we give optimal error estimates for the third order fully discrete IMEX-UWDG scheme. Numerical results are given in Section 5 to verify the main theoretical results. In Section 6, we give some concluding remarks.