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A Diagonalization-Based Parallel-in-Time Algorithm for Crank-Nicolson's Discretization of the Viscoelastic Equation

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Abstract. In this paper, we extend a diagonalization-based parallel-in-time (PinT) algorithm to the viscoelastic equation. The central difference method is used for spatial discretization, while for temporal discretization, we use the Crank-Nicolson scheme. Then an all-at-once system collecting all the solutions at each time level is formed and solved using a fixed point iteration preconditioned by an α-circulant matrix in parallel. Via a rigorous analysis, we find that the spectral radius of the iteration matrix is uniformly bounded by $\alpha/(1-\alpha)$, independent of the model parameters (the damping coefficient ε and the wave velocity $\sqrt{\gamma}$) and the discretization parameters (the time step τ and the spatial mesh size h). Unlike the classical wave equation with Dirichlet boundary condition where the upper bound $\alpha/(1-\alpha)$ is very sharp, we find that the occurrence of the damping term $-\varepsilon\Delta y_t$, as well as the large final time T, leads to even faster convergence of the algorithm, especially when α is not very small. We illustrate our theoretical findings with several numerical examples.

AMS subject classifications: 65F08, 65F10, 65M22, 65Y05

Key words: Parallel-in-time (PinT) algorithm, Crank-Nicolson method, diagonalization, viscoelastic equation, convergence analysis.

1. Introduction

In this paper, we consider the following viscoelastic equation:

$$\begin{aligned} &y_{tt}-\varepsilon\Delta y_t-\gamma\Delta y=f, && (\mathbf{x},t)\in\Omega\times J,\\ &y(\mathbf{x},t)=\phi(\mathbf{x},t), && (\mathbf{x},t)\in\partial\Omega\times J,\\ &y(\mathbf{x},0)=\psi_1(\mathbf{x}), && \mathbf{x}\in\Omega,\\ &y_t(\mathbf{x},0)=\psi_2(\mathbf{x}), && \mathbf{x}\in\Omega, \end{aligned} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ is an open domain, $J = (0, T]$ is the time interval, $\varepsilon \geq 0$ is the damping coefficient and $\sqrt{\gamma}$ is the wave velocity with $\gamma > 0$. The source function f, the boundary function $\phi(\mathbf{x}, t)$ and the initial functions $\psi_1(\mathbf{x})$ and $\psi_2(\mathbf{x})$ are all given. Without loss of generality, we assume that a homogeneous boundary condition is applied, i.e. $\phi(\mathbf{x}, t) = 0$, to simplify the theoretical analysis.

Comparing with the classical wave equation, the viscoelastic equation uses a damping term $-\varepsilon \Delta y_t$ to provide a more accurate model in many applications e.g. in the propagation of vibration waves through viscoelastic media [26]. We refer the reader to [50] for more applications in science and engineering. Recently, Gander *et al.* [14] numerically validated that the viscoelastic damping $-\varepsilon \Delta y_t$ works much better than the first order damping term $-y_t$ (which results in a telegrapher's equation) for modeling the vibration of an elastic string.

The well-posedness of the viscoelastic equation had been addressed a long time ago [2, 39, 42]. However, like most PDEs in applications, it is very hard to find the analytical solutions due to the complicated given data, or, the complex defining domain. As a result, it is very essential to study the numerical methods for solving the viscoelastic equation. In fact, all prevalent techniques for spatial discretization can be applied to the viscoelastic equation. See, for example, [9, 45] for finite element methods, [3, 17, 25, 38] for mixed finite element methods, [20, 54] for finite difference methods, [22, 23] for generalized finite difference methods (also known as finite volume element methods), [43] for discontinuous Galerkin methods and [49] for weak Galerkin finite element methods. Recently, the meshless methods also attract much attention in solving the viscoelastic equation (1.1), see for instance [35, 37].

However, most numerical methods for solving Eq. (1.1) are based on time stepping. For example, the reader can refer to [30, 53] for a Crank-Nicolson scheme, where an extrapolation approach and a proper orthogonal decomposition technique are used to reduce the degree of freedom. To speedup the computation, one can apply parallel computing technique at each time level due to the fact that the time discretization leads to a steady partial differential equation. Amongst the various parallel computing methods, the domain decomposition method attracts the most attention. The domain decomposition method originated from Schwarz's seminal work [41] in 1870, and was developed in 1990 by Lions [28] as a parallel solver, which finally leads to the optimized Schwarz methods — cf. [10] and references therein. Other efficient domain decomposition methods include the restricted additive Schwarz (RAS) method [47], the finite element tearing and interconnecting (FETI) method [8], the balancing domain decomposition by constraints (BDDC) [6], etc. We refer the reader to the monographs [7, 47] for many other variants of the aforementioned methods in detail. An alternative to the domain decomposition method would be the multigrid method, cf. for example [48]. Obviously, these methods can be straightforwardly applied to the viscoelastic equation (1.1) after time discretization.

In order to use computing resources more efficiently, one can apply the parallel-in-time (PinT) computation to further accelerate the solution of the viscoelastic equation (1.1). In fact, Adey and Brebbia [1] proposed long ago to solve the viscoelastic equation in parallel in the Laplace transformed plane by a finite element method, and then a least square colloca-