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A POSTERIORI ERROR ESTIMATES FOR DARCY-FORCHHEIMER'S PROBLEM COUPLED WITH THE CONVECTION-DIFFUSION-REACTION EQUATION

FAOUZI TRIKI, TONI SAYAH*, AND GEORGES SEMAAN

Abstract. In this work we derive *a posteriori* error estimates for the convection-diffusion-reaction equation coupled with the Darcy-Forchheimer problem by a nonlinear external source depending on the concentration of the fluid. We introduce the variational formulation associated to the problem, and discretize it by using the finite element method. We prove optimal *a posteriori* errors with two types of calculable error indicators. The first one is linked to the linearization and the second one to the discretization. Then we find upper and lower error bounds under additional regularity assumptions on the exact solutions. Finally, numerical computations are performed to show the effectiveness of the obtained error indicators.

Key words. Darcy-Forchheimer problem, convection-diffusion-reaction equation, finite element method, *a posteriori* error estimates.

1. Introduction.

This work deals with the *a posteriori* error estimate of the Darcy-Forccheimer system coupled with the convection-diffusion-reaction equation. We consider following system of equations:

$$(P)\begin{cases} \dfrac{\mu}{\rho}K^{-1}\mathbf{u}+\dfrac{\beta}{\rho}|\mathbf{u}|\mathbf{u}+\nabla p &= \mathbf{f}(\cdot,C) \quad \text{in}\,\Omega,\\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in}\,\Omega,\\ -\alpha\Delta C+\mathbf{u}\cdot\nabla C+r_0C &= g \quad \text{in}\,\Omega,\\ \mathbf{u}\cdot\mathbf{n} &= 0 \quad \text{on}\,\Gamma,\\ C &= 0 \quad \text{on}\,\Gamma, \end{cases}$$

where $\Omega\subset\mathbb{R}^d$, $d=2,3$, is a bounded simply-connected open domain, having a Lipschitz-continuous boundary Γ with an outer unit normal $\mathbf{n}$. The unknowns are the velocity $\mathbf{u}$, the pressure p and the concentration C of the fluid. $|.|$ denotes the Euclidean norm, $|\mathbf{u}|^2=\mathbf{u}\cdot\mathbf{u}$. The parameters ρ, μ and β represent the density of the fluid, its viscosity and its dynamic viscosity, respectively. β is also referred as Forchheimer number when it is a scalar positive constant. The diffusion coefficient α and the parameter r_0 are strictly positive constants. The function $\mathbf{f}$ represents an external force that depends on the concentration C and the function g represents an external concentration source. K is the permeability tensor, assumed to be uniformly positive definite (i.e $x^TKx>0$ for all $x\in R^d\setminus\{\mathbf{0}\}$) and bounded such that there exist two positive real numbers K_m and K_M such that

$$0<K_m\le\left\|K^{-1}\right\|_{L^\infty(\Omega)^{d\times d}}\le K_M. \tag{1}$$

It is important to note that K_m should be smaller than the smallest eigenvalue of K^{-1} over Ω and K_M could be very large.

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*Corresponding author.

System (P) represents the coupling of the Darcy-Forchheimer problem with the convection-diffusion-reaction equation satisfied by the concentration of the fluid. The same system can represent the coupling of the Darcy-Forchheimer system with the heat equation by replacing the concentration C by the temperature T and setting $r_0 = 0$.

Darcy's law (see [31] and [42] for the theoretical derivation) is an equation that describes the flow of a fluid through a porous medium. This law was formulated by Darcy based on experimental results. It is simply the first equation of the system (P) where the dynamic viscosity $\beta = 0$. In the case where the velocity of the fluid is higher and the porosity is non uniform, Forchheimer proposed the Darcy-Forchheimer equation (see [22]) which is the first equation of system (P) by adding the non-linear term). Several numerical and theoretical studies of the Darcy-Forchheimer equation were performed, and among others we mention [25, 27, 32, 28, 33].

For the coupling of Darcy's equation with the heat equation, we refer to [9] where the system is treated using a spectral method. The authors in [7] and [16] considered the same stationary system but coupled with a nonlinear viscosity that depends on the temperature. In [17], the authors derived an optimal *a posteriori* error estimate for each of the numerical schemes proposed in [7]. We can also refer to [3] where the authors used a vertex-centred finite volume method to discretize the coupled system. For physical applications of system (P), we refer to [39]. In [36], we introduced the variational formulation associated to system (P), and we showed uniqueness under additional constraints on the concentration. Then, we discretized the system by using the finite element method and we showed the existence and uniqueness of corresponding solutions. Moreover, we established the *a priori* error estimate between the exact and numerical solutions and introduced a numerical scheme where we studied the corresponding convergence.

I. Babuška was the first who introduced *a posteriori* analysis (see [4]), then it was developed by R. Verfürth [41], and has been the object of a large number of publications. Many works have established the *a posteriori* error estimates for the Darcy flow, see for instance [2, 10, 11, 29]. In [17], the authors established *a posteriori* error estimates for Darcy's problem coupled with the heat equation. Sayah T. (see [34]) established the *a posteriori* error estimates for the Brinkman-Darcy-Forchheimer problem. Moreover, in [35], we established the *a posteriori* estimates for the Darcy-Forchheimer problem without the convection-diffusion-reaction equation. Furthermore, several works established the *a priori* and *a posteriori* errors for the time-dependent convection-diffusion-reaction equation coupled with Darcy's equation(see [12, 13]).

The main goal of this work is to derive the *a posteriori* error estimates associated to the coupling system (P) for the numerical scheme introduced in [36]. We start by recalling some auxiliary results from [36] concerning the discretization of system (P), and the numerical scheme with the corresponding convergence. In a second step, we establish the *a posteriori* error estimates where the error between the exact and iterative numerical solutions are bounded by two types of local indicators: the indicators of discretization and the indicators of linearization. Then, we show the corresponding efficiency by bounding each indicator by the local error. Finally, we present some numerical computations in order to show the effectiveness of the proposed method.

The outline of the paper is as follows:

- Section 2 is devoted to the continuous problem.