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SEMI-CONVERGENCE OF AN ALTERNATING-DIRECTION ITERATIVE METHOD FOR SINGULAR SADDLE POINT PROBLEMS

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Abstract. For large-scale sparse saddle point problems, Peng and Li [12] have recently proposed a new alternating-direction iterative method for solving nonsingular saddle point problems, which is more competitive (in terms of iteration steps and CPU time) than some classical iterative methods such as Uzawa-type and HSS (Hermitian skew splitting) methods. In this paper, we further study this method when it is applied to the solution of singular saddle point problems and prove that it is semi-convergent under suitable conditions.

Key words. saddle point problem, semi-convergence, singular, alternating-direction iterative, iterative method

1. Introduction

Consider the saddle point problem

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix} \text{ or } \tilde{A}\tilde{x} = b, \tag{1.1}$$

where the matrices $A \in R^{m\times m}$ and $B \in R^{m\times n}$ and the vectors $f \in R^m$ and $g \in R^n$ are given, with $n \leq m$, and B^T is the transpose of the matrix B. We shall assume that A is positive real (i.e., $A^T + A$ is positive definite) and B is a rectangular matrix with rank$(B) = r$. Saddle point problems arise in many scientific computing and engineering applications such as mixed finite element methods for solving elliptic partial differential equations, and Stokes problems, computational fluid dynamics, and constrained least-squares problems; see [7, 8, 9, 11], for example. Benzi et al. [5] gave a comprehensive survey for recent work on the saddle point problems.

When the matrix B is of full column rank, i.e., $r = n$, we know that the coefficient matrix of system (1.1) is nonsingular and this sytem has a unique solution. Because the matrices A and B are usually large and sparse, iterative methods are always considered to be the most suitable candidates for solving system (1.1). So far, a large variety of iterative methods based on the matrix splitting of the coefficient matrix of (1.1) have been studied in the literature, for example, Uzawa-type methods [4, 6], GSSOR (generalized symmetric successive over-relaxation) iterative methods [1, 10], and HSS iterative methods [3]. Recently, Peng and Li [12] has proposed a new alternating-direction iterative method for solving system (1.1). Theoretical analysis and numerical experiments have shown that this new iterative method is more competitive than some classical iterative methods in terms of iteration steps and CPU time, such as the Uzawa-type and HSS iterative methods with optimal parameters.

When $r < n$, the coefficient matrix of system (1.1) is singular and this system has an infinite number of solutions; see [5]. Recently, Zheng et al. [14] has showed that

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the GSOR (generalized successive over-relaxation) iterative method proposed in [4] can be used to solve a singular saddle point problem of type (1.1) and proved that this method is semi-convergent. In this paper, we will study the new alternating-direction iterative (ADI) method presented in [12] when it is applied to the solution of the singular system (1.1), and prove that this new method is semi-convergent under suitable conditions.

2. An Alternating-direction Iterative Method

In this section, we review the new alternating-direction iterative method proposed in [12] for solving the nonsingular saddle point problem (1.1).

Define

$$\tilde{A} = \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} 0 & 0 \\ -B^T & 0 \end{pmatrix}.$$

We then consider the following splittings of $\tilde{A}$:

$$\tilde{A} = (\alpha_1 I + \tilde{H}) - (\alpha_1 I - \tilde{S}) = (\alpha_2 I + \tilde{S}) - (\alpha_2 I - \tilde{H}),$$

where I denotes the identity matrix with the corresponding dimension and α_1 and α_2 are positive parameters.

Now, given the initial guess $\tilde{x}^{(0)}$, a new-ADI method can be described as follows:

$$\begin{cases} (\alpha_1 I + \tilde{H})\tilde{x}^{k+\frac{1}{2}} = (\alpha_1 I - \tilde{S})\tilde{x}^k + b, \\ (\alpha_2 I + \tilde{S})\tilde{x}^{k+1} = (\alpha_2 I - \tilde{H})\tilde{x}^{k+\frac{1}{2}} + b. \end{cases}$$

By eliminating the intermediate vector $\tilde{x}^{(k+\frac{1}{2})}$, this method can be equivalently rewritten as

$$\tilde{x}^{(k+1)} = T_{\alpha_1,\alpha_2}\tilde{x}^{(k)} + c,$$

where

$$\begin{aligned} T_{\alpha_1,\alpha_2} &= (\alpha_2 I + \tilde{S})^{-1}(\alpha_2 I - \tilde{H})(\alpha_1 I + \tilde{H})^{-1}(\alpha_1 I - \tilde{S}), \\ c &= (\alpha_2 I + \tilde{S})^{-1}[I + (\alpha_2 I - \tilde{H})(\alpha_1 I + \tilde{H})^{-1}]b. \end{aligned} \tag{2.1}$$

Lemma 2.1. *With the above definition, the matrix* T_{α_1,α_2} *has the form*

$$T_{\alpha_1,\alpha_2} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \tag{2.2}$$

where

$$\begin{aligned} T_{11} &= \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}(\alpha_1 I + A)^{-1}(\alpha_1^2 I - BB^T) - \frac{\alpha_1}{\alpha_2}I, \\ T_{12} &= -\frac{\alpha_1+\alpha_2}{\alpha_2}(\alpha_1 I + A)^{-1}B, \\ T_{21} &= \frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2^2}B^T(\alpha_1 I + A)^{-1}(\alpha_1^2 I - BB^T) + \frac{\alpha_2^2-\alpha_1^1}{\alpha_1\alpha_2^2}B^T, \\ T_{22} &= I - \frac{\alpha_1+\alpha_2}{\alpha_2^2}B^T(\alpha_1 I + A)^{-1}B. \end{aligned}$$

Proof. Note that

$$(\alpha_1 I + \tilde{H})^{-1}(\alpha_2 I - \tilde{H}) = (\alpha_2 I - \tilde{H})(\alpha_1 I + \tilde{H})^{-1},$$

which is true, since each term in this equation is a polynomial of $\tilde{H}$. Then, with the splitting $\tilde{A} = M - N$, where M is a nonsingular matrix, T_{α_1,α_2} is the corresponding iteration matrix

$$T_{\alpha_1,\alpha_2} = M^{-1}N,$$