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On Linear Homogeneous Biwave Equations

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Abstract. The biwave maps are a class of fourth order hyperbolic equations. In this paper, we are interested in the solution formulas of the linear homogeneous biwave equations. Based on the solution formulas and the weighted energy estimate, we can obtain the $L^\infty(\mathbb{R}^n)-W^{N,1}(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)-W^{N,2}(\mathbb{R}^n)$ estimates, respectively. By our results, we find that the biwave maps enjoy some different properties compared with the standard wave equations.

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1 Introduction

Let $u:\mathbb{R}^{1+n}\to N$ is a smooth map from a Minkowski space $\mathbb{R}^{1+n}$ into a Riemannian manifold N, then the bi-energy functional is given by

$$\begin{aligned} E(u)=&\frac{1}{2}\int_{\mathbb{R}^{1+n}}||(d+d^*)^2u||^2\mathrm{d}t\mathrm{d}x=\frac{1}{2}\int_{\mathbb{R}^{1+n}}||d^*du||^2\mathrm{d}t\mathrm{d}x\\ =&\frac{1}{2}\int_{\mathbb{R}^{1+n}}||\tau_\Box(u)||^2\mathrm{d}t\mathrm{d}x. \end{aligned}\tag{1.1}$$

Here

$$\tau_\Box^\alpha(u)=\Box u^\alpha+\Gamma^\alpha_{\beta\gamma}(-u_t^\beta u_t^\gamma+\sum_{i=1}^n u_i^\beta u_i^\gamma),$$

where $\Box=\frac{\partial^2}{\partial t^2}-\triangle_x$ is the wave operator on $\mathbb{R}^{1+n}$ and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of N. Clearly, the map u is a wave map iff the wave field $\tau_\Box^\alpha(u)$ vanishes.

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The Euler-Lagrange equation of (1.1) is

$$\begin{aligned}(\tau_2)_\square(u)^\alpha =& J_u(\tau_\square u)^\alpha = \triangle\tau_\square(u)^\alpha + R^{N\alpha}(df,df)\tau_\square(u)\\ =&\square\tau_\square(u)^\alpha + \Gamma^\alpha_{\mu\gamma}\Big(-\tau_\square(u)^\mu_t\tau_\square(u)^\gamma_t + \sum_{i=1}^n \tau_\square(u)^\mu_i\tau_\square(u)^\gamma_i\Big)\\ &+R^\alpha_{\beta\gamma\mu}\Big(-u^\beta_t u^\gamma_t + \sum_{i=1}^n u^\beta_i u^\gamma_i\Big)\tau_\square(u)^\mu = 0, \end{aligned} \tag{1.2}$$

i.e., $\tau_\square(u)$ is a Jacobi field, where $R^\alpha_{\beta\gamma\mu}$ is the Riemannian curvature of N, see [1].

Biwave maps are biharmonic maps on Minkowski space, which generalize wave maps, and have been first studied by Chiang [2–4] in 2009. Biwave maps satisfy a fourth order hyperbolic system of partial differential equations, which are different from biharmonic maps. Recently, Chiang and Wei [1] studied the long time behavior of the biwave maps by Klainerman's method of vector fields when the initial data are small.

In this paper, for simplicity, in order to study the well posedness of biwave maps, we first study the linear case. If we assume that the manifold is flat, which means $\Gamma^\alpha_{\mu\gamma}=0$ and $R^\alpha_{\beta\gamma\mu}=0$, we will obtain $\square^2 u=0$.

Then the Cauchy problem for the n-dimensional biwave equation satisfies the following system

$$\begin{cases}\square^2 u=0,\\ u(x,0)=f_1(x),\\ u_t(x,0)=f_2(x),\\ u_{tt}(x,0)=f_3(x),\\ u_{ttt}(x,0)=f_4(x).\end{cases} \tag{1.3}$$

Here $u=u(x,t)$ is the unknown, $t>0$ and $x\in\mathbb{R}^n$, $u_t=\frac{\partial u}{\partial t}$, $f_i(x)\in C_0^\infty(\mathbb{R}^n)$, $i=1,\cdots,4$. Occasionally it is convenient to write $t=x^0$, in which case $\partial_0=\partial_t$.

The biwave equations are also related to the mathematical theory of elasticity [5], thus it is of great physical significance to study it. However, there are not many mathematical results on biwave equations, because it gets more difficult when studying the high-order PDEs. Feng and Neilan [6,7] develop the finite element methods for the approximations of biwave equations. Fushchych, Roman and Zhdanov [8] consider the symmetry analysis of biwave equations $\square^2 u=F(u)$ and obtain the exact solution by Ansätze invariants under the non-conjugate subalgebras of the extended Poincaré algebra and the conformal algebra. The existence and uniqueness of the solution to initial-value problem and boundary-value problem for the fourth-order linear PDE of hyperbolic and composite types are given by Korzyuk, Cheb and Konopel'ko [9,10], respectively. By the techniques of Fourier analysis, Korzyuk, Vinh and Minh [5] also get the solution formulas for the Cauchy problem of the $n-$dimensional biwave maps $(\frac{\partial^2}{\partial t^2}-a^2\triangle)(\frac{\partial^2}{\partial t^2}-b^2\triangle)u=0$, with $a^2\neq b^2$.