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Energy-Preserving Hybrid Asymptotic Augmented Finite Volume Methods for Nonlinear Degenerate Wave Equations

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Dedicated to Professor Yirang Yuan for his 90th birthday

Abstract. In this paper we develop and analyze two energy-preserving hybrid asymptotic augmented finite volume methods on uniform grids for nonlinear weakly degenerate and strongly degenerate wave equations. In order to deal with the degeneracy, we introduce an intermediate point to divide the whole domain into singular subdomain and regular subdomain. Then Puiseux series asymptotic technique is used in singular subdomain and augmented finite volume scheme is used in regular subdomain. The keys of the method are the recovery of Puiseux series in singular subdomain and the appropriate combination of singular and regular subdomain by means of augmented variables associated with the singularity. Although the effect of singularity on the calculation domain is conquered by the Puiseux series reconstruction technique, it also brings difficulties to the theoretical analysis. Based on the idea of staggered grid, we overcome the difficulties arising from the augmented variables related to singularity for the construction of conservation scheme. The discrete energy conservation and convergence of the two energy-preserving methods are demonstrated successfully. The advantages of the proposed methods are the energy conservation and the global convergence order determined by the regular subdomain scheme. Numerical examples on weakly degenerate and strongly degenerate under different nonlinear functions are provided to demonstrate the validity and conservation of the proposed method. Specially, the conservation of discrete energy is also ensured by using the proposed methods for both the generalized Sine-Gordeon equation and the coefficient blow-up problem.

AMS subject classifications: 35L05, 35L80, 41A60, 65M08, 68W30
Key words: Nonlinear degenerate wave equations, energy conservations, Puiseux series, augmented variable, augmented finite volume method.

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1 Introduction

The challenging degenerate partial differential equations can be used to describe practical problems [4, 9], such as seawater desalination, motion of liquids and gases in porous media, phenomena in plasma, etc., and also appear in physics [20], economics [10], population genetics [2], climatology [7], engineering [42, 49], cloaking [18], vision [12] and other applications [16, 26, 43]. Nonlinear wave equations are known to be more powerful than their linear analogues in describing the propagation of waves in many physical applications, such as the Sine-Gordon equation and the nonlinear wave equation with exponential nonlinearities. In addition, in non-homogeneous media this can lead to nonlinear wave equations with variable coefficients, for example, variable coefficients changes in permittivity or changes in refractive index, which affect wave propagation [21]. And the loss of uniform ellipticity of the variable coefficient results in nonlinear degenerate wave equations. In particular, the nonlinear degenerate wave equation can be used to describe the propagation of long waves (e.g. tsunami waves) generated by local sources and their rise on the coast. This is due to the fact that in describing beach rise, the wave equation is only given in the domain bounded by the shoreline, with the velocity vanishing at the boundary, thus making the wave equation degenerate. The nonlinear degenerate wave equations studied in this paper are precisely such degenerate problems at the boundary.

In this paper, we consider the nonlinear degenerate wave equation

$$u_{tt}-(x^\alpha u_x)_x+f(u)=0, \quad (x,t)\in Q. \tag{1.1}$$

Here, $Q=(0,b)\times(0,T)$, and $b>0$, $T>0$ and $0<\alpha<2$ are real constants. It is obvious that equation (1.1) is degenerate at a single point inside the space domain. Eq. (1.1) can be used to describe the run-up of waves on a sloping beach in tsunami wave theory [15], the transverse wave of a string under tension [3], and the wave equation with a non-homogeneous wave speed [19], etc. We rewrite the equation (1.1) as the following system of two equations by introducing the auxiliary quantity $v=u_t$

$$u_t=v, \quad v_t-(x^\alpha u_x)_x+f(u)=0, \quad (x,t)\in Q. \tag{1.2}$$

Furthermore, for the different value of α, the degeneracy of equation (1.1) is divided into weakly degenerate and strongly degenerate. For $t\in(0,T)$, the boundary condition is

$$u(0,t)=u(b,t)=0, \quad \text{for } 0<\alpha<1, \text{ (weakly degenerate)}, \tag{1.3}$$

$$\left(x^\alpha\frac{\partial u}{\partial x}\right)(0,t)=u(b,t)=0, \quad \text{for } 1\le\alpha<2, \text{ (strongly degenerate)}. \tag{1.4}$$

For both cases, we have the following initial conditions

$$u(x,0)=\varphi(x), \quad u_t(x,0)=\psi(x), \quad x\in(0,b). \tag{1.5}$$

In [50], the authors proved the well-posedness of the linear wave equation under the weakly degenerate case and the strongly degenerate case, and it was shown that the well-posedness of the nonlinear degenerate wave equation was similarly provable in [51].