Journal of Nonlinear Modeling and Analysis
Volume 6, Number 1, March 2024, 56–70

http://jnma-online.com
DOI:10.12150/jnma.2024.56

Positive Solutions for Third Order Three-Point Boundary Value Problems with p-Laplacian*

Xingfang Feng[1,2†], Hanying Feng[3]

Abstract In this paper, the existence of positive solutions of the following third-order three-point boundary value problem with p-Laplacian

$$\begin{cases} (\phi_p(u''(t)))' + f(t,u(t)) = 0,\ t \in (0,1), \\ u(0) = \alpha u(\eta), u(1) = \alpha u(\eta), u''(0) = 0, \end{cases}$$

is studied, where $\phi_p(s) = |s|^{p-2}s,\ p > 1$. By using the fixed point index method, we establish sufficient conditions for the existence of at least one or at least two positive solutions for the above boundary value problem. The main result is demonstrated by providing an example as an application.

Keywords Positive solution, three-point boundary value problem, fixed point index, p-Laplacian operator

MSC(2010) 34A08.

1. Introduction

The purpose of this paper is to study the existence of positive solutions for the following third-order three-point boundary value problem (BVP for short) with p-Laplacian

$$(\phi_p(u''(t)))' + f(t,u(t)) = 0,\ t \in (0,1), \tag{1.1}$$

$$u(0) = \alpha u(\eta), u(1) = \alpha u(\eta), u''(0) = 0, \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $0 < \alpha,\ \eta < 1$.

There has been an extensive study on boundary value problems with diverse boundary conditions via many methods [1, 11, 21]. The equation with p-Laplacian operator arises in the modeling of different physical and natural phenomena, non-Newtonian mechanics [4, 10], combustion theory [18], population biology [16, 17] and nonlinear flow laws [5, 13]. Therefore, there exist a very lager number of

†The corresponding author.
Email address: fxfg651@163.com (X. Feng), fhanying@126.com (H. Feng)
[1]Department of Mathematics, Hebei Normal University, Shijiazhang 050024, P. R. China
[2]Department of Scientific Culture, Shijiazhuang Branch, Army Engineering University of PLA, Shijiazhuang 050003, Hebei, P. R. China
[3]Department of Mathematics, Nantong Institute of Technology, Nantong 226002, P. R. China
*The authors were supported by the foundation for Basic Disciplines of Army Engineering University of PLA ((Grant No.KYSZJQZL2013) and Nantong Institute of Technology Natural Science Fund (NITNSF) (Grant No. 201824).

papers devoted to the existence of solutions to the p-Laplacian boundary value problems with various boundary conditions, which have been studied by many authors [7, 14, 15, 19, 20] and references therein. Iyase [8] proved the existence of solutions for a third-order multipoint boundary value problem at resonance by using the coincidence degree arguments. Additionally, Iyase and Imaga [9] applied Leray-Schauder continuation principle to establish at least one solution to the third-order p-Laplacian boundary value problem.

Recently, Li [12] has studied the existence of positive solutions for the third-order boundary value problem with p-Laplacian operator.

$$\begin{aligned} &(\phi_p(u''(t)))' + f(t, u(t), u'(t), u''(t)) = 0,\ t \in (0, 1),\\ &au(0) - bu'(0) = 0,\ cu(1) + du'(1) = 0,\ u''(0) = 0, \end{aligned} \tag{1.3}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. By using the fixed point theorem of Krasnosel'skii, the author established the existence results for positive solutions to (1.3).

Motivated by the above works, our purpose here is to give existence of positive solutions for a third-order three-point boundary value problem with p-Laplacian operator. In this paper, we construct a Green function and study its properties, and then transform BVP (1.1) and (1.2) into an equivalent integral equation. Next, applying the fixed point index theorem, we establish the existence of at least one or at least two positive solutions for the above boundary value problem. For convenience, we list the following assumptions:

(H_1) $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous;
(H_2) $0 < \alpha,\ \eta < 1$.

2. Preliminaries and several important lemmas

In this section, we provide some basic concepts and properties of fixed point index for compact maps.

Let $E = C[0, 1]$, $C^+[0, 1] = \{x \in C[0, 1] |\ x(t) \geq 0,\ t \in [0, 1]\}$, then E is a Banach space with norm $\|u\| = \max_{t\in[0,1]} |u(t)|$.

Definition 2.1.([6]) Let E be a real Banach space. Let P be a nonempty, convex closed set in E. We say that P is a cone if it satisfies the following properties:
(i) $\lambda u \in P$ for $u \in P, \lambda \geq 0$;
(ii) $u, -u \in P$ implies $u = \theta$ (θ denotes the null element of E).

If $P \subset E$ is a cone, we denote the order induced by P on E by $\leq$. For $u, v \in P$, we write $u \leq v$ if and only if $v - u \in P$.

Lemma 2.1. *Assume that (H_1) holds and $\alpha \neq 1$. Then for any $x \in C^+[0, 1]$, the problem*

$$(\phi_p(u''(t)))' + f(t, x(t)) = 0,\ t \in (0, 1), \tag{2.1}$$

$$u(0) = \alpha u(\eta), u(1) = \alpha u(\eta), u''(0) = 0 \tag{2.2}$$

has the unique solution

$$u(t) = \int_0^1 G(t, s)\phi_q(\int_0^s f(\tau, x(\tau))\mathrm{d}\tau)\mathrm{d}s + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta, s)\phi_q(\int_0^s f(\tau, x(\tau))\mathrm{d}\tau)\mathrm{d}s, \tag{2.3}$$