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Existence of Three Weak Solutions for a Class of Quasi-Linear Elliptic Operators with a Mixed Boundary Value Problem Containing $p(\cdot)$-Laplacian in a Variable Exponent Sobolev Space

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Abstract In this paper, we consider a mixed boundary value problem to a class of nonlinear operators containing $p(\cdot)$-Laplacian. More precisely, we are concerned with the problem with the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show the existence of at least three weak solutions under some hypotheses on given functions and the values of parameters.

Keywords $p(\cdot)$-Laplacian type equation, three weak solutions, mixed boundary value problem

MSC(2010) 35H30, 35D05, 35J60, 35J70.

1. Introduction

In this paper, we consider the following nonlinear problem

$$\begin{cases} -\mathrm{div}\,[S_t(x,|\boldsymbol{\nabla} u(x)|^2)\boldsymbol{\nabla} u(x)] = \lambda f_0(x,u(x)) + \mu f_1(x,u(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Gamma_1, \\ S_t(x,|\boldsymbol{\nabla} u(x)|^2)\frac{\partial u}{\partial \boldsymbol{n}}(x) = \lambda g_0(x,u(x)) + \mu g_1(x,u(x)) & \text{on } \Gamma_2, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ is a bounded domain with a Lipschitz-continuous boundary Γ, Γ_1 and Γ_2 are disjoint open subsets of Γ such that $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma$, and $\boldsymbol{n}$ denotes the unit, outer and normal vector to Γ. Thus, we impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given data $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g_i : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ for $i = 0, 1$ are Carathéodory functions, and λ, μ are parameters. The function $S(x,t)$ is a Carathéodory function on $\Omega \times [0,\infty)$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$ and $S_t = \partial S/\partial t$. Then, the function $\mathrm{div}\,[S_t(x,|\boldsymbol{\nabla} u(x)|^2)\boldsymbol{\nabla} u(x)]$ is a more general operator than the $p(\cdot)$-Laplacian $\Delta_{p(x)}u(x) = \mathrm{div}\,[|\boldsymbol{\nabla} u(x)|^{p(x)-2}\boldsymbol{\nabla} u(x)]$, where $p(x) > 1$. This generality brings about difficulties and requires more conditions.

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The study of such a type of differential equations with $p(\cdot)$-growth conditions has been a very interesting topic recently. Studying such a problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [32]) and electrorheological fluids (Diening [12], Halsey [19], Mihăilescu and Rădulescu [24] as well as Růžička [26]).

Over the last two decades, there have been many articles on the existence of weak solutions for the Dirichlet boundary condition for the $p(\cdot)$-Laplacian type, that is,

$$\begin{cases} -\mathrm{div}\,[|\nabla u|^{p(x)-2}\nabla u] = f(x,u) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Gamma. \end{cases} \tag{1.2}$$

See, Fan [14], Ji [21,22], Fan and Zhang [16], Avci [7] and Yücedağ̂ [28], for example. On the other hand, for the Steklov boundary condition, that is,

$$\begin{cases} -\mathrm{div}\,[|\nabla u|^{p(x)-2}\nabla u] = f(x,u) & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma. \end{cases} \tag{1.3}$$

See, Fan and Ji [15], Wei and Chen [27], Yücedağ̂ [29], Allaoui et al., [1], Ayoujil [8] and Deng [11], for example.

However, since we can find only a few papers on the problem with the mixed boundary value condition in variable exponent Sobolev space as in (1.1) (cf. Aramaki [4–6]), we are convinced of the reason for existence of this paper.

Throughout this paper, we assume that Γ_1 and Γ_2 are disjoint open subsets of Γ such that

$$\overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma \text{ and } \Gamma_1 \neq \emptyset. \tag{1.4}$$

When $p(x) = p = \text{const.}$, Zeidler [30] considered the following mixed boundary value problem

$$\begin{cases} \mathrm{div}\,\boldsymbol{j} = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_1, \\ \boldsymbol{j}\cdot\boldsymbol{n} = h & \text{on } \Gamma_2, \end{cases} \tag{1.5}$$

where $\boldsymbol{j}$ is the current density, and $f(x), g(x)$ and $h(x)$ are given functions. If $\boldsymbol{j}$ is of the form

$$\boldsymbol{j} = -\alpha(|\nabla u|^2)\nabla u, \tag{1.6}$$

problem (1.5) corresponds to many physical problems, for example, hydrodynamics, gas dynamics, electrostatics, heat conduction, elasticity and plasticity.

If $\alpha \equiv 1$, then problem (1.5) becomes

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial \boldsymbol{n}} = h & \text{on } \Gamma_2. \end{cases} \tag{1.7}$$

From the mathematical point of view, this is a mixed boundary value problem for the Poisson equation. If $\alpha(|\nabla u|^2) = |\nabla u|^{p-2}$, problem (1.5) corresponds to the p-Laplacian equation. Definitely, if $\Gamma_2 = \emptyset$ (resp. $\Gamma_1 = \emptyset$), then system (1.5) becomes