Journal of Computational Mathematics
Vol.42, No.4, 2024, 911–931.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2212-m2021-0231

A COUPLED METHOD COMBINING CROUZEIX-RAVIART NONCONFORMING AND NODE CONFORMING FINITE ELEMENT SPACES FOR BIOT CONSOLIDATION MODEL*

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Abstract

A mixed finite element method is presented for the Biot consolidation problem in poroelasticity. More precisely, the displacement is approximated by using the Crouzeix-Raviart nonconforming finite elements, while the fluid pressure is approximated by using the node conforming finite elements. The well-posedness of the fully discrete scheme is established, and a corresponding priori error estimate with optimal order in the energy norm is also derived. Numerical experiments are provided to validate the theoretical results.

Mathematics subject classification: 65N15, 65M60, 74F10.
Key words: Poroelasticity, Nonconforming finite element, Inf-sup condition, A priori error estimate.

1. Introduction

Poroelasticity describes the interaction between a fluid flow and a deformable elastic porous medium that is saturated in the fluid. Its theoretical basis was initially established by Biot [4]. Biot poroelastic theory can be found in a wide range of applications. For example, the mathematical models for land subsidence in geomechanics [11, 45], carbon sequestration [43, 44, 46], safe long-term disposal of wastes in environment engineering [43, 46], and brain edema [29], are all poroelastic models.

In the context of numerical treatments for poroelastic equations, finite element methods (FEMs) are the most commonly used approaches [40, 41]. It is well known that a direct continuous Galerkin approximation may cause Poisson locking or nonphysical pressure oscillations (we call these two phenomenons as poroelaticity locking) [21, 42–44, 46]. In the literature, there have been some developed numerical methods to avoid the poroelaticity locking. We review those which motivated our work: A two-field formulation of the Biot model with the elastic displacement and pressure being unknowns was approximated by using the MINI element with

* Received August 10, 2021 / Revised version received February 22, 2022 / Accepted December 20, 2022 / Published online May 31, 2023 /
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a stabilization term [48]. With the same formulation, discontinuous Galerkin (DG) methods, weak Galerkin methods and hybrid high order methods were developed in [5, 13, 14, 26, 52]. By introducing an additional fluid flux variable, a new three-field formulation was obtained, and a numerical method that couples DG methods and mixed FEMs was investigated in [45]. Later, this method was extended to nonconforming FEMs [27, 36, 54], virtual element methods [51], weak Galerkin methods [50], locally conservative DG methods [24, 30, 56]. Some numerical schemes based on four-field formulation can be found in [33, 35, 55]. Interested readers can also refer to [10, 11, 19, 31, 32] for other study of locking-free numerical methods for the Biot model. Moreover, for the corresponding fast solvers, some preconditioners which are robust with respect to the physical and discretization parameters have been developed in [1, 3, 12, 25, 37]. As mentioned above, the key issue in the numerical solution of the Biot problem is to address two numerical instabilities: Possion locking phenomenon and nonphysical pressure oscillations. In the two-field formulation these two instabilities have been initially discussed in the earlier works [40–42] from the regularity point of view. Therein they utilized inf-sup stabilized Stokes FEMs to discretize equations. However, recently it is shown in [2,48] that only inf-sup condition is not enough to overcome nonphysical pressure oscillations. They suggested adding a time-dependent stabilization term to address this issue. On the other hand, in the case of three-field formulation, the authors in [46] combine numerical tests with heuristic analysis to explain that the poroelasticity locking typically occurs when the storage coefficient is very close to zero and the small time step is used. In such a case, the poroelasticity model behaves as an incompressible model. For a remedy, they suggested using DG scheme or nonconforming FEM to approximate the displacement variable [45]. Though many works have been devoted to addressing these issues, most of them only numerically verify whether the constructed elements are locking-free or pressure-oscillations free, the corresponding mathematical justifications and interpretations behind these methods are relatively rare. More recently, an interesting result in [24] concluded that, in the three-field formulation, when the FE pairs for the displacements, Darcy velocity, and pore pressure satisfy suitable compatibilities, one can obtain a family of parameter-robust numerical schemes. In the same formulation, another breakthrough in the theoretical aspect can be found in a recent paper [55], in which the regularity of the Biot model is firstly obtained, the cause of pressure oscillations is reinvestigated from an algebraic viewpoint, and the Poisson locking is viewed from the classical FE approximation of linear elasticity. Moreover, based on the theoretical analysis, the author of [55] developed a new three-field mixed FEM that is free of locking and nonphysical pressure oscillations. The objective of the present work is to design a robust two-field FEM for the Biot model. We shall use the nonconforming Crouzeix-Raviart (CR) [15] finite element to approximate the elastic displacement, and adopt the standard linear continuous finite element for the pore pressure. It is shown that the error order estimate is robust with respect to the Lamé constant λ (more details can be found in Remark 5.3 in Section 5). In other words, we give a mathematical analysis that explains why nonconforming CR FEM displacement discretization can overcome Possion locking in poroelasticity. In addition, motivated by [48], a stabilization term is imposed in order to remove the nonphysical pressure oscillations arising from the continuous FE approximation. The finite element pair used in this article was initially proposed and analyzed for Stokes problems [38], in which a stabilization was suggested. Later, Lamichhane pointed out that the stabilization proposed in [38] is unnecessary [34]. It means that such a finite element pair satisfies the inf-sup condition for Stokes equations. For a two-dimensional linear elasticity problem, a CR finite element was introduced to overcome the locking phenomenon in [9]. The corresponding three-dimensional case was