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EFFICIENT SPECTRAL METHODS FOR EIGENVALUE PROBLEMS OF THE INTEGRAL FRACTIONAL LAPLACIAN ON A BALL OF ANY DIMENSION*

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Abstract

An efficient spectral-Galerkin method for eigenvalue problems of the integral fractional Laplacian on a unit ball of any dimension is proposed in this paper. The symmetric positive definite linear system is retained explicitly which plays an important role in the numerical analysis. And a sharp estimate on the algebraic system's condition number is established which behaves as N^{4s} with respect to the polynomial degree N, where $2s$ is the fractional derivative order. The regularity estimate of solutions to source problems of the fractional Laplacian in arbitrary dimensions is firstly investigated in weighted Sobolev spaces. Then the regularity of eigenfunctions of the fractional Laplacian eigenvalue problem is readily derived. Meanwhile, rigorous error estimates of the eigenvalues and eigenvectors are obtained. Numerical experiments are presented to demonstrate the accuracy and efficiency and to validate the theoretical results.

Mathematics subject classification: 65N35, 65N25.
Key words: Integral fractional Laplacian, Spectral method, Eigenvalue problem, Regularity analysis, Error estimate.

1. Introduction

Nonlocal operators have been an active area of research in different branches of mathematics. These operators arise in many applications such as image processing, finance, electromagnetic fluids, peridynamics, and porous media flow [7, 13, 23, 24, 33, 36], among which the fractional Laplace operator is of common interests of mathematicians and physicists.

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In this work, we consider eigenvalue problems concerning the integral fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$ on the unit ball of any dimension

$$\begin{cases} (-\Delta)^s u = \lambda u, & \boldsymbol{x} \in \mathbb{B}^d, \\ u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \mathbb{R}^d \setminus \mathbb{B}^d. \end{cases} \tag{1.1}$$

Here the fractional Laplacian is defined in singular integral [4]

$$(-\Delta)^s u(\boldsymbol{x}) = C(d,s) \int_{\mathbb{R}^d} \frac{u(\boldsymbol{x}) - u(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^{d+2s}} \mathrm{d}\boldsymbol{y}, \quad C(d,s) = \frac{2^{2s} s \Gamma(s+d/2)}{\pi^{d/2}\Gamma(1-s)}, \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{1.2}$$

It can also be equivalently defined via a pseudodifferential operator of symbol $\|\boldsymbol{\xi}\|^{2s}$ in the Fourier space. Indeed, for a function u of the Schwartz class,

$$(-\Delta)^s u(\boldsymbol{x}) = \left[\mathscr{F}^{-1}\left(\|\boldsymbol{\xi}\|^{2s} \widehat{u}(\boldsymbol{\xi})\right)\right](\boldsymbol{x}), \tag{1.3}$$

where we denote by $\mathscr{F} f$ or simply by $\widehat{f}$ the Fourier transform of any function $f(\boldsymbol{x}) \in L^2(\mathbb{R}^d)$, and denote by $\mathscr{F}^{-1}\widehat{f}$ the inversion of the Fourier transform

$$\begin{aligned} \widehat{f}(\boldsymbol{\xi}) &= [\mathscr{F} f](\boldsymbol{\xi}) := \int_{\mathbb{R}^d} f(\boldsymbol{x}) \mathrm{e}^{-\mathrm{i}\langle \boldsymbol{\xi}, \boldsymbol{x}\rangle} \mathrm{d}\boldsymbol{x}, \\ f(\boldsymbol{x}) &= \left[\mathscr{F}^{-1}\widehat{f}\,\right](\boldsymbol{x}) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\boldsymbol{\xi}) \mathrm{e}^{\mathrm{i}\langle \boldsymbol{\xi}, \boldsymbol{x}\rangle} \mathrm{d}\boldsymbol{\xi}. \end{aligned}$$

This eigenvalue problem is closely related to fractional quantum mechanics such as the fractional Schrödinger equation. In this regard, eigenfunctions of the fractional Laplacian correspond to the energy states of the system being modeled [31]. Many researchers have shown their interest in this kind of fractional problem from the physical, mathematical and computational point of view. Most of existing studies focus on the theoretical research [9, 15, 19, 22, 26]. Kwaśnicki [30] introduced the Weyl's asymptotic law for the eigenvalues of the one-dimensional fractional Laplace operator $(-\Delta)^s$ on the interval $(-1,1)$ with the zeros exterior boundary conditions: The n-th eigenvalue λ_n is equal to $(n\pi/2 - (2-2s)\pi/8)^{2s} + \mathcal{O}(1/n)$. Chen *et al.* [14] and DeBlassie [17] have derived the estimate for the n-th eigenvalue λ_n on a bounded convex domain in $\mathbb{R}^d$ is $(n\pi/2)^{2s}/2 \le \lambda_n \le (n\pi/2)^{2s}$. Owing to the non-locality of the fractional Laplacian, it is usually impossible to obtain analytically a closed expression for the eigenfunctions, and it is also hard to precisely specify the behavior of an eigenfunction near the boundary of the unit ball. This motivates researchers to carry out numerical studies on eigenvalue problems of the fractional Laplacian. Borthagaray *et al.* [10] have studied the finite element approximation for one- and two-dimensional eigenvalue problems of the fractional Laplacian in which it showed the eigenfuncions belonged to $H^{s+1/2-\varepsilon}$ for any $\varepsilon > 0$, and the conforming finite element method exhibited a convergence rate of order $1-\varepsilon$. As mentioned above, the eigenfunctions of the fractional Laplacian operator have only a limited regularity measured in usual Sobolev space, and eigensolutions obtained by ordinary numerical methods have a very poor accuracy. Thus, high order methods may be required to conquer this difficulty.

Some recent advances have been gained on related theoretical analysis and numerical computations on the fractional differential equations [1, 18, 28, 29, 34, 38, 40, 42, 43]. However, the regularity estimates of solutions to the high-dimensional fractional diffusion-reaction equation are not yet available. Acosta *et al.* [2] presented the regularity of $(1-x^2)^{-s}u(x)$ was $r+2s$ when the regularity index for the right hand side function was r in weighted Sobolev spaces