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Real Reduced Matrix mKdV Integrable Hierarchies Under Two Local Group Reductions

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Abstract. We propose a kind of reduced Ablowitz-Kaup-Newell-Segur matrix spectral problems under two local group reductions by similarity transformations. Associated integrable hierarchies of matrix mKdV type integrable models are presented, which amend the complex matrix mKdV integrable hierarchies. Zero curvature equations are key objects in generating integrable models.

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1. Introduction

The zero curvature formulation provides a systematical scheme to generate integrable models [6]. The key is to choose a matrix spectral problem and then an associated hierarchy of integrable models can be computed via zero curvature equations. The inverse scattering transform exactly uses the matrix spectral problem to solve Cauchy problems of integrable models, the evolution of the scattering data being determined by the associated temporal matrix spectral problems [4].

Matrix spectral problems with free potentials are standard and natural. But reduced matrix spectral problems are more restrictive and harder to apply. A idea of using similarity transformations is adopted for formulating reduced matrix spectral problems, which lead to integrable hierarchies (see, e.g., [14]). The aim of using similarity transformations is to make it easier to achieve to keep the corresponding zero curvature equations invariant

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and so generate integrable models. Two such typical kinds of integrable models are the nonlinear Schrödinger equations and the modified Korteweg-de Vries equation. Both of them are generated from the Ablowitz-Kaup-Newell-Segur (AKNS) matrix spectral problems by taking one similarity transformation. Moreover, taking a pair of similarity transformations can engender more diverse integrable models. Some difficulty might be involved, since two reductions on potentials, corresponding to the pair of similarity transformations, bring new requirements on balancing associated zero curvature equations.

Very recently, the idea of taking similarity transformations has also been applied to construction of nonlocal integrable models [3]. Three kinds of reduced integrable nonlinear Schrödinger type equations, and two kinds of reduced integrable modified Kortweweg-de Vries type equations have been proposed and classified [16]. The inverse scattering transform has also been developed to solve nonlocal integrable models (see, e.g., [2,10,15,22]). There are other efficient approaches which attempt nonlocal integrable models, and particularly, construct soliton solutions. The Hirota bilinear method, Darboux transformation, Bäclund transforms and the Riemann-Hilbert technique have been proved to be powerful and many theories have been proposed for different reduced integrable models, both local and non-local (see, for example, [7–9, 14, 26, 33]).

In this paper, we would like to propose a pair of local group reductions by similarity transformations for the AKNS matrix spectral problems to generate reduced integrable models. The rest of the paper is organized as follows. In Section 2, we recall the AKNS matrix spectral problems and their associated hierarchies of matrix integrable models to prepare the subsequent analyses. In Section 3, we consider two local group reductions by similarity transformations for the AKNS matrix spectral problems simultaneously and generate reduced local hierarchies of real matrix mKdV integrable models. In Section 4, we illustrate the presented formulation with concrete examples, which present abundant reduced AKNS matrix spectral problems and reduced corresponding matrix integrable models, including novel mKdV type integrable models. In the last section, we summarize the results and give some concluding remarks.

2. The Standard AKNS Matrix Integrable Hierarchies

Let $m, n \geq 1$ be two arbitrarily given natural numbers. For each pair of $m, n \geq 1$, we state the AKNS matrix spectral problems and the associated AKNS hierarchies of matrix integrable models, to facilitate the subsequent analyses.

First, we denote the spectral parameter by λ, and assume that p and q are two submatrix potentials

$$p = p(x,t) = (p_{jk})_{m\times n}, \quad q = q(x,t) = (q_{kj})_{n\times m}. \tag{2.1}$$

The standard matrix AKNS spectral problems reads

$$-i\phi_x = U\phi, \quad U = U(u,\lambda) = (\lambda\Lambda + P), \tag{2.2}$$

and

$$-i\phi_t = V^{[r]}\phi, \quad V^{[r]} = V^{[r]}(u,\lambda) = \left(\lambda^r\Omega + Q^{[r]}\right), \quad r \geq 0, \tag{2.3}$$