INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 5, Number 4, Pages 299–316

BABUŠKA'S PENALTY METHOD FOR INHOMOGENEOUS DIRICHLET PROBLEM: ERROR ESTIMATES AND MULTIGRID ALGORITHMS

THIRUPATHI GUDI

Abstract. This article is two fold. Firstly, we derive optimal order *a priori* error estimates for Babuška's penalty method applied to inhomogeneous Dirichlet problem. Secondly, we derive convergence of W-cycle and V-cycle multigrid algorithms for the resulting system. To this end, a simple pre-conditioner is introduced to remedy the ill-condition due to over-penalty.

Key words. Finite element, multigrid, penalty, pre-conditioner.

1. Introduction

We consider the model problem of finding $u \in H^1(\Omega)$ such that $u|_{\partial\Omega} = g$ and

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in H_0^1(\Omega). \tag{1}$$

Here $f \in L_2(\Omega)$ and $g \in C^0(\partial\Omega)$ is such that there is a $\psi \in H^2(\Omega)$ with $\psi|_{\partial\Omega} = g$. We consider $\Omega \subset \mathbb{R}^2$ to be a bounded convex polygonal domain. Let the boundary $\partial\Omega = \cup_{i=1}^N \bar{\Gamma}_i$, where Γ_is are pairwise disjoint (open) line segments on $\partial\Omega$. From the regularity theory of elliptic problems [18], it holds that $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \le C \left(\|f\| + \sum_{i=1}^N \|g\|_{H^{3/2}(\Gamma_i)} \right), \tag{2}$$

hereafter $\|\cdot\|$ denotes the standard $L_2(\Omega)$ norm. Based on (1), it is well known that the finite element method with interpolated boundary condition is defined as follows: Find $u_h \in V_h$ with $u_h|_{\partial\Omega} = g_h$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \forall v \in V_h^0, \tag{3}$$

where g_h is an approximation of g and $V_h^0 \subset H_0^1(\Omega)$ is a finite element subspace. In practice, g_h is considered to be a nodal interpolation of g in $V_h|_{\partial\Omega}$, where $V_h \subset H^1(\Omega)$ is a finite element subspace. However it is shown in [4] that the error estimates for (3) are better when g_h is chosen to be the L_2-projection of g onto $V_h|_{\partial\Omega}$. In particular when g_h is chosen to be a nodal interpolation, the L_2 error estimate requires g to be a piecewise H^2 function on $\partial\Omega$, see [4, Theorem 7.1]. Perhaps this estimate may not be improved. On the other-hand a mild disadvantage associated with the choice that g_h is L_2-projection, we need to solve a system before we solve for the solution u_h.

An alternative method due to Babuška [1] is to pose the boundary condition weakly by penalty. Namely, find $u_h \in V_h$ such that

$$a_h(u_h, v) = L_h(v) \quad \forall v \in V_h, \tag{4}$$

Received by the editors October 12, 2014.
1991 *Mathematics Subject Classification.* 65N30, 65N15.
The author is supported partly by the DST Fast Track Project and partly by the UGC Center for Advanced Study.

where

$$a_h(w,v) = \int_\Omega \nabla w \cdot \nabla v \, dx + \int_{\partial\Omega} \frac{1}{h^2} wv \, ds, \tag{5}$$

and

$$L_h(v) = \int_\Omega f v \, dx + \int_{\partial\Omega} \frac{1}{h^2} gv \, ds. \tag{6}$$

In [1], error estimates that are almost optimal order up to an arbitrary ϵ were derived for homogeneous Dirichlet boundary value problem. In this article, we improve upon the error estimates in [1] and derive optimal order error estimates for general inhomogeneous Dirichlet problem. To this end, we construct an interpolation which is partially based on a projection. We derive optimal order approximation properties for this interpolation and use as a key in our *a priori* error analysis. On the other hand since the method in (4) uses high penalty on posing the boundary condition weakly, the condition number results in the order h^{-3}. We construct a simple diagonal pre-conditioner to remedy this ill-condition and present W-cycle and V-cycle multigrid algorithms for the resulting preconditioned system. We prove optimal order convergence for the multigrid algorithms following the theory developed in [2, 6, 7, 20, 21, 11, 12] for W-cycle algorithm and the additive multigrid theory in [8] for V-cycle algorithm. The convergence analysis of V-cycle algorithm is established under a mild assumption on the initial triangulation of the domain. Namely, we require that each interior vertex in the initial mesh is connected by at most six edges. The results in this article will be useful in solving the feedback boundary control problems (see e.g. [17]) and also in the analysis of over penalized discontinuous Galerkin and non-conforming methods. There is a plenty of work on multigrid methods for finite element methods, we refer to a few articles and monographs for the references and related work [3, 5, 14, 15, 19, 23, 24, 25, 26].

Remark 1.1. It is well known that there is another alternative method due to Nitsche [22] posing the Dirichlet boundary conditions weakly. However this approach requires tuning of a suitable penalty parameter for obtaining a stable formulation. We therefore restrict ourself in this article to the penalty method by Babuška.

The rest of the article is organized as follows. In section 2, we present improved error estimates for the penalty method. In section 3, we construct a simple pre-conditioner and set up multigrid methods. In sections 4 and 5, we present convergence analysis of W-cycle and V-cycle algorithms, respectively. In section 6, we present some numerical experiments demonstrating the performance of the method and multigrid algorithms. Finally, we conclude the article in section 7.

2. A priori Analysis of Penalty Method

In this section, we derive optimal order *a priori* error estimates for the penalty method (4). Let $\mathcal{T}_h$ be a regular quasi-uniform simplicial triangulation of Ω. Let $h_T = \sqrt{|T|}$, where $|T|$ is the area of T. Set $h = \max\{h_T : T \in \mathcal{T}_h\}$. The P_1 finite element space V_h is defined as

$$V_h = \{v \in C^0(\bar{\Omega}) : v|_{T \in \mathcal{T}_h} \in P_1(T)\}.$$

We also need the following finite element space:

$$V_h^0 = \{v \in V_h : v|_{\partial\Omega} = 0\}.$$