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UNIFORM CONVERGENCE VIA PRECONDITIONING

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Abstract. The linear singularly perturbed convection-diffusion problem in one dimension is considered and its discretization on the Shishkin mesh is analyzed. A new, conceptually simple proof of pointwise convergence uniform in the perturbation parameter is provided. The proof is based on the preconditioning of the discrete system.

Key words. singular perturbation, convection-diffusion, boundary-value problem, Shishkin mesh, finite differences, uniform convergence, preconditioning.

1. Introduction

We consider the following one-dimensional singularly perturbed problem of convection-diffusion type,

$$\mathcal{L}u := -\varepsilon u'' - b(x)u' + c(x)u = f(x),\ x \in (0,1),\ u(0) = u(1) = 0, \tag{1}$$

with a small positive perturbation parameter ε and $C^1[0,1]$-functions b, c, and f, where b and c satisfy

$$b(x) \geq \beta > 0, \quad c(x) \geq 0 \quad \text{for } x \in I := [0,1].$$

It is well known, see [6, 9] for instance, that (1) has a unique solution u in $C^3(I)$, which in general has an exponential boundary layer near $x = 0$.

Singular perturbation problems arise in various applications, see [3, 4]. Typical of them are boundary and/or interior layers, regions whose size decreases as $\varepsilon \to 0$ and where the solution changes abruptly. This is why these problems require special numerical methods [5, 10, 4, 12, 7]. One of the most popular methods is to use an appropriate finite-difference scheme on the layer-adapted meshes of Shishkin [10, 4, 12, 7] or Bakhvalov [13, 12, 7] types.

We consider here the standard upwind discretization of (1) on the Shishkin mesh with N mesh steps. It is shown in [11] that for the matrix of the resulting system the condition number in the maximum norm is of magnitude $\mathcal{O}(\varepsilon^{-1}(N/\ln N)^2)$. Since this is unsatisfactory when $\varepsilon \to 0$, a simple preconditioning is proposed in the same paper. This behavior of the condition number is contrasted in [11] to that of the singularly perturbed reaction-diffusion problem, which can be described as (1) with $b \equiv 0$ and $c > 0$ on I. When the reaction-diffusion problem is discretized using the standard central scheme on the Shishkin mesh, there is no need for preconditioning because the condition number behaves like $\mathcal{O}((N/\ln N)^2)$.

We note that there is another difference between the two types of the singularly perturbed problems, viz. the difference in the proofs of ε-uniform convergence of the numerical solution to the discretized continuous solution. One of the ways to prove that a finite-difference discretization yields ε-uniform convergence is to use the following principle, which originated from non-perturbed problems (cf. [2, 13, 5]):

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Principle 1. *ε-uniform stability and ε-uniform consistency imply ε-uniform convergence.*

Moreover, ε-uniform pointwise convergence is desired when solving singular perturbation problems. For reaction-diffusion problems, this can be achieved by using the following version ([13]) of the above principle:

Principle 2. *ε-uniform stability and ε-uniform consistency, both in the maximum norm, imply ε-uniform pointwise convergence.*

However, Principle 2 does not work for convection-diffusion problems (1) because ε-uniform pointwise consistency is not present, although it is easy to show that the upwind scheme is ε-uniformly stable in the maximum norm. For these problems, ε-uniform consistency can be proved in a discrete L^1 norm and this is why the proofs based on Principle 1 have to rely on some kind of hybrid stability inequality [5, 1, 8, 7], an approach that typically involves the discrete Green's function. Other ε-uniform convergence proofs also exist, like those that use barrier functions [10, 4, 12, 7].

Our main result is that we show that essentially the same preconditioning (we appropriately modify the method from [11]), which eliminates the difference in the condition numbers of simple finite-difference discretizations for the convection-diffusion and reaction-diffusion problems, can also be used to eliminate the difference in the proofs of ε-uniform pointwise convergence for these two problem types. In other words, a suitable preconditioning technique enables the use of Principle 2 for the convection-diffusion problem. Using this approach, we prove an almost (up to logarithmic factors) first-order pointwise ε-uniform convergence for the upwind scheme discretizing the problem (1) on the Shishkin mesh. This result, however, is not the main contribution of this paper, because the same has already been proved elsewhere (see the above references). Rather, we feel that the main contribution is this conceptually simple proof which points out that there is a connection between conditioning and ε-uniform pointwise convergence for convection-diffusion problems.

The rest of the paper is organized as follows. We give the properties of the continuous solution in Section 2. Then, in Section 3, we introduce the finite-difference scheme on the Shishkin mesh and discuss the conditioning of the discrete problem. Section 4 provides the proof of ε-uniform pointwise convergence. Finally, some concluding remarks are given in Section 5.

2. Properties of the continuous solution

The solution u of (1) can be decomposed into the smooth and boundary-layer parts. We present here Linß's [7, Theorem 3.48] version of such a decomposition:

$$u(x) = s(x) + y(x), \tag{2}$$

$$|s^{(k)}(x)| \le C\left(1 + \varepsilon^{2-k}\right), \quad |y^{(k)}(x)| \le C\varepsilon^{-k}e^{-\beta x/\varepsilon}, \tag{3}$$
$$x \in I, \quad k = 0, 1, 2, 3.$$

Above and throughout the paper, C denotes a generic positive constant which is independent of ε. For the construction of the function s, see [7], since the details are not of interest here. As for y, it solves the problem

$$\mathcal{L}y(x) = 0, \quad x \in (0,1), \quad y(0) = -s(0), \quad y(1) = 0. \tag{4}$$

It is important to note that y satisfies a homogeneous differential equation. We shall use this fact later on in the paper.