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OPTIMAL ORDER CONVERGENCE IMPLIES NUMERICAL SMOOTHNESS

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Abstract. It is natural to expect the following loosely stated approximation principle to hold: a numerical approximation solution should be in some sense as smooth as its target exact solution in order to have optimal convergence. For piecewise polynomials, that means we have to at least maintain numerical smoothness in the interiors as well as across the interfaces of cells or elements. In this paper we give clear definitions of numerical smoothness that address the across-interface smoothness in terms of *scaled* jumps in derivatives [9] and the interior numerical smoothness in terms of differences in derivative values. Furthermore, we prove rigorously that the principle can be simply stated as *numerical smoothness is necessary for optimal order convergence.* It is valid on quasi-uniform meshes by triangles and quadrilaterals in two dimensions and by tetrahedrons and hexahedrons in three dimensions. With this validation we can justify, among other things, incorporation of this principle in creating adaptive numerical approximation for the solution of PDEs or ODEs, especially in designing proper smoothness indicators or detecting potential non-convergence and instability.

Key words. Adaptive algorithm, discontinuous Galerkin, numerical smoothness, optimal order convergence.

1. Introduction

Consider the problem of approximating a function u defined on a domain in $\mathbb{R}^n$ by a sequence of numerical solutions $\{u_h\}$. The target function u may be an exact solution of a second or higher order partial or ordinary differential equation, and the sequence may be piecewise polynomials from a discontinuous Galerkin method [7] or reconstructed polynomials u^R in an intermediate phase [8], and even post-processed finite element solutions to achieve superconvergence [14]. Although we had discontinuous Galerkin numerical solutions in mind, the source of the problem is not important for our purpose here, as it only puts the degree of smoothness of u in perspective. Now suppose that u is in $W_s^{p+1}(\Omega)$ (standard notation for Sobolev spaces here, supindex for the order of derivative and subindex for the L^s based space). It is natural to expect that the approximation solutions should be as smooth (in some sense) in order to achieve optimal convergence rate. The purpose of this paper is to give clear and rigorous results on this simple minded principle.

Sun [9] showed in one dimension if the mesh is uniform and the function u has $p+1$ weak derivatives in $L^s, s = 1, 2, \infty$, then a necessary condition can be formulated. In particular in the $s = \infty$ case, the jumps of the k^{th} derivatives, (across a mesh point) of the approximation piecewise polynomial of degree p must be less than or equal to $\mathcal{O}(h^{p+1-k}), 0 \le k \le p$. This one dimensional result is perhaps not surprising, once one realizes the interpolation error behaves in a similar way: taking a derivative, one loses a power of h, assuming $u \in W_\infty^{p+1}$. In the appendix of this paper, the assertion is actually proved by comparing the derivatives of u, its continuous piecewise Lagrange interpolant u^I, and u^R at a mesh point. This short proof can even be carried over to higher dimensions. Unfortunately, it cannot be

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extended to higher dimensions when $s = 1, 2$ due to the restriction on continuity imposed by the Sobolev imbedding theorem (See Remark 4.1 in the appendix for other reasons). Since now one starts with a function $u \in W_s^{p+1}, s = 1, 2$, there are always some k and up for which the kth derivative of u at a point of interest is not well defined. On the other hand, in hindsight an idea (Lemma 2.1 below) in the much lengthier and originally unfavored proof in [9] for one dimension can be distilled and generalized to prove the two and three dimensional versions of the same principle.

While Sun *et al.* [11, 12] have successfully applied it to the analysis of numerical methods for one dimensional nonlinear conservation laws, it is quite clear that this principle has a very broad scope of applications such as safeguarding divergence or negating optimal order convergence in designing new methods, let alone in creating smoothness indicators [11, 12] in an adaptive algorithm, and so on. Being motivated by its application potential in higher dimensions, in this paper we generalize the concept of numerical smoothness of a piecewise polynomial in [9] to higher dimensions and show that in order for the convergence of u_h to u to have optimal order p in W_s^{p+1}, u_h must have $W_s^{p+1}, s = 1, 2, \infty$ numerical smoothness, provided that the domain can be meshed by quasi-uniform subdivisions into triangles or quadrilaterals in 2-D and tetrahedrons or hexahedrons (cubes) in 3-D. We accounted for both interior and across interface numerical smoothness. In § 3, we formerly define the across-interface numerical smoothness in Definition 3.1, which is well motivated by the theorems in § 2 and also define interior numerical smoothness in Definition 3.2. The main result that states optimal order convergence implies numerical smoothness is proved in Theorems 3.3 and 3.4. This section is written in such a way that the reader can go read it directly after the introduction section.

The organization of rest of this paper is as follows. In § 2, we first derive basic error estimates without imposing conditions on meshes other than the shape regularity. The main theorem is Theorem 2.11, now under the quasi-uniform condition on the mesh. Finally, in § 4 we give a short proof of the one dimensional version of Theorem 3.3 and explain why it cannot be extended to higher dimensions.

2. Basic Estimates for Numerical Smoothness

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n), \alpha_i \geq 0, 1 \leq i \leq n$ be a multi-index and $|\alpha| = \sum_i^n \alpha_i$. Some of the theorems in this section have their one dimensional counterparts in [9]. We are especially inspired by the central use of Lemma 2.2 in [9]. The next lemma is its higher dimensional version, which will be used after a scaling argument back to the master element of unity size. At a certain point $x \in \mathbb{R}^n$ of interest, e.g., a mesh nodal point, a center of a simplex (edge or face), to measure the smoothness of a mesh function u_h, we will be examining all the jumps $[\![\partial^\alpha u_h]\!]_x, |\alpha| = k$ in the partial derivatives of order k for $0 \leq k \leq p$. In this perspective, we now state and prove the next lemma. Denote by $\mathbb{P}_p$ the space of polynomials of total degree at most p.

Lemma 2.1. *Let $\Delta = (\Delta_0, \Delta_1, \cdots, \Delta_p)$, where each Δ_k is a vector of a certain length (e.g., it has as many components as the number of partial derivatives of order k). Let $\hat{\Omega}_\pm$ be two open sets in $\mathbb{R}^n$. Define*

$$Q(\Delta) = \min_{\hat{v} \in \mathcal{P}} \left(\left\| \hat{v} + \frac{1}{2} \sum_{\alpha \in \mathcal{I}} \frac{\Delta_\alpha}{\alpha!} \xi^\alpha \right\|^2_{L^2(\hat{\Omega}_-)} + \left\| \hat{v} - \frac{1}{2} \sum_{\alpha \in \mathcal{I}} \frac{\Delta_\alpha}{\alpha!} \xi^\alpha \right\|^2_{L^2(\hat{\Omega}_+)} \right),$$