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A Priori Error Estimates of Finite Element Methods for Linear Parabolic Integro-Differential Optimal Control Problems

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Abstract. In this paper, we study the mathematical formulation for an optimal control problem governed by a linear parabolic integro-differential equation and present the optimality conditions. We then set up its weak formulation and the finite element approximation scheme. Based on these we derive the a priori error estimates for its finite element approximation both in H^1 and L^2 norms. Furthermore some numerical tests are presented to verify the theoretical results.

AMS subject classifications: 65N30, 65R20

Key words: Optimal control, linear parabolic integro-differential equations, optimality conditions, finite element methods, a priori error estimate.

1 Introduction

Optimal control problems governed by partial differential equations have been a major research topic in applied mathematics and control theory. Since the milestone work of J. P. Lions [10], a great deal of progress has been made in many aspects like stability, observability and numerical methods, which are too extensive to be mentioned here even very briefly. Among them, finite element approximations of optimal control problems governed by various partial differential equations, either linear or nonlinear, have been

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much studied in the literature. For instance, optimal control problems governed by linear elliptic PDEs have been much studied, and their finite element approximation, and a prior error estimates were established in [3]. Many results in convergence of the standard finite element approximation of optimal control problems governed by linear or nonlinear elliptic and parabolic equations can be found in, for examples, [1,3,4,17–19,22–24], although it is impossible to give even a very brief review here. Recently, optimal control problems with more complicated state equations have been considered, particularly those with the integro-differential state equations, which are often met in real applications. For example, progress on the finite element method for the optimal control problem governed by elliptic integral equations and integro-differential equations has been made in [8], in which the a priori and a posteriori error estimations were obtained.

Parabolic integro-differential equations and their control are often met in applications such as heat conduction in materials with memory, population dynamics, and viscous-elasticity, cf. e.g., Friedman and Shinbrot [5], Heard [7], and Renardy, Hrusa and Nohel [20]. For equations with nonsmooth kernels, we refer to Grimmer and Pritchard [6], Lunardi and Sinestrari [12], and Lorenzi and Sinestrari [13] and references therein. Furthermore finite element methods for parabolic integro-differential equations problems with a smooth kernel have been discussed in, e.g., Cannon and Lin [2], LeRoux and Thomée [14], Lin, Thomée, and Wahlbin [15], Sloan and Thomée [21], Thomée and Zhang [25], and Yanik and Fairweather [27].

However there exists little research on optimal control problems governed by parabolic integro-differential equations, in spite of the fact that such control problems are widely encountered in practical engineering applications and scientific computations. Furthermore the finite element method of this optimal control problem governed by such equations is not well-studied although there exists much research on the finite element approximation of parabolic integro-differential equations as mentioned above. Those will be studied in this work with numerical verifications.

The content of the paper is as follows. In Section 2, we present the weak formulation and analyze the existence of the solution for the optimal control problem. In Section 3, we give the optimality conditions and the finite element approximation of the optimal control problems. In Section 4, we establish the a priori error estimates for the finite element approximation of the control problem. In the last section, we perform some numerical tests, which illustrate the theoretical results.

Throughout the paper, we adopt the standard notations for Sobolev spaces as in [9–11, 26], such as $W^{m,q}(\Omega)$ on Ω with norm $\|\cdot\|_{m,q,\Omega}$, and semi-norm $|\cdot|_{m,q,\Omega}$ for $1\leq q\leq\infty$. Set $W_0^{m,q}(\Omega)=\{w\in W^{m,q}(\Omega): w|_{\partial\Omega}=0\}$. Also denote $W^{m,2}(\Omega)(W_0^{m,2}(\Omega))$ by $H^m(\Omega)$ $(H_0^m(\Omega))$, with norm $\|\cdot\|_{m,\Omega}$, and semi-norm $|\cdot|_{m,\Omega}$. Denote by $L^s(0,T;W^{m,q}(\Omega))$ the Banach space of all L^s integrable functions from $(0,T)$ into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(0,T;W^{m,q}(\Omega))}=(\int_0^T\|v\|^s_{W^{m,q}(\Omega)}dt)^{1/s}$ for $s\in[1,\infty)$ and the standard modification for $s=\infty$. Similarly, one can define the spaces $H^1(0,T;W^{m,q}(\Omega))$ and $C^k(0,T;W^{m,q}(\Omega))$. The details can be found in [11].