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HIGHER-ORDER LINEARIZED MULTISTEP FINITE DIFFERENCE METHODS FOR NON-FICKIAN DELAY REACTION-DIFFUSION EQUATIONS

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Abstract. In this paper, two types of higher-order linearized multistep finite difference schemes are proposed to solve non-Fickian delay reaction-diffusion equations. For the first scheme, the equations are discretized based on the backward differentiation formulas in time and compact finite difference approximations in space. The global convergence of the scheme is proved rigorously with convergence order $\mathcal{O}(\tau^2+h^4)$ in the maximum norm. Next, a linearized noncompact multistep finite difference scheme is presented and the corresponding error estimate is established. Finally, extensive numerical examples are carried out to demonstrate the accuracy and efficiency of the schemes, and some comparisons with the implicit Euler scheme in the literature are presented to show the effectiveness of our schemes.

Key words. Non-Fickian delay reaction-diffusion equation, linearized compact/noncompact, multistep finite difference scheme, solvability, convergence.

1. Introduction

Nonlinear delay partial differential equations (NDPDEs) are widely used in description of natural phenomena and social behaviors in biology, medicine, control theory, epidemiology, climate models, and many others [6, 16, 20, 39, 45]. These equations have been paid a lot of attention because they provide a powerful tool to reflect the essential characteristics of processes with delayed effects. However, the analytical solutions of most of the delay partial differential equations (DPDEs) can not be explicitly expressed and the theoretical analysis of DPDEs is also difficultly carried out because of the delayed terms. Hence, developing efficient and higher-order numerical methods for DPDEs especial NDPDEs has become an important issue and a hot topic [9, 19, 42, 43].

In this paper, we are dedicated to developing the higher-order numerical approximation to the solution of non-Fickian delay reaction-diffusion equation of the form

$$\frac{\partial u}{\partial t}=D_1\frac{\partial^2 u}{\partial x^2}+\frac{D_2}{\delta}\int_0^t e^{-\frac{t-w}{\delta}}\frac{\partial^2 u}{\partial x^2}(x,w)dw+f(u(x,t),u(x,t-s),x,t), \tag{1}$$

where $(x,t)\in[a,b]\times[0,T]$, D_1, D_2 and δ are positive constants, and $s>0$ is the delay parameter. The initial condition associated with (1) is given by

$$u(x,t)=\varphi(x,t),\ x\in[a,b],\ t\in[-s,0] \tag{2}$$

and the boundary conditions are specified by

$$u(a,t)=u_a(t),\ u(b,t)=u_b(t),\ t>0. \tag{3}$$

Equation (1) is called non-Fickian delay reaction-diffusion equation due to certain memory effects taken into account [6, 20]. In the case of $D_2=0$, it reduces to a

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regular delayed reaction-diffusion equation, which we frequently encounter in a vast array of fields. For example, if we take

$$f(u(x,t),u(x,t-s),x,t) = -au(x,t) + \frac{bu(x,t-s)}{1+u^m(x,t-s)},$$

then, we reduce the equation (1) to the diffusive Mackey-Glass equation [19, 20], and if we take

$$f(u(x,t),u(x,t-s),x,t) = -au(x,t) + bu(x,t-s)e^{-u^m(x,t-s)},$$

then, we obtain the diffusive Nicholson's blowflies equation [10,21,22,25–29,32,33].

As a typical partial integro-differential equation, the equation (1) has been paid more attention and extensively studied [1–4,6–8,11–15,18,20,23,24,30,35,38,40,41]. In 1986, Sloan *et al* [34] numerically studied it by the backward Euler and Crank-Nicolson methods. In [13], Fedotov presented an asymptotic method for the analysis of traveling waves in a one-dimensional reaction-diffusion system where the diffusion has a finite velocity with Kolmogorov-Petrovskii-Piskunov kinetics. Araújo [5] investigated the qualitative properties of numerical traveling wave solutions for integro-differential equations. Recently, Khuri *et al* [18] concentrated on the finite difference method and the spline collocation method for the numerical solution of a generalized Fisher integro-differential equation, and Branco *et al* [6] studied the structure of the solution to the non-Fickian delay reaction-diffusion equations from both the theoretical and numerical points of view. In [44], Zhang *et al* constructed a second-order linearized finite difference scheme for the generalized Fisher-Kolmogorov-Petrovskii-Piskunov equation by introducing a new variable which transforms the integro-differential equation into an equivalent coupled system of first-order differential equations. Late then, Kazem [17] considered a meshless method on non-Fickian flows with mixing length growth in porous media based on radial basis functions. Very recently, Li *et al* [20] discussed the long time behavior of non-Fickian delay reaction-diffusion equations and Wang [38] analyzed the finite element method for fully discrete semilinear evolution equations with positive memory based on two-grid discretizations.

However, most of the numerical methods are no more than second-order accuracy, while there are a large number of scenarios where higher-order accurate schemes are a necessity due to the desired accuracy of the simulations. On the other hand, the higher-order schemes allow one to approximate a solution with fewer grid points, while maintaining the same accuracy as a low-order scheme. In certain circumstances, the desired grid point size is based on the ability to resolve the structure of the solution, and not on the accuracy of computation. But, the higher-order finite-difference schemes, typically achieved by computing derivatives with a wider matrix stencil, cause some difficulties near the boundary, just as one must be able to calculate the inner point near the boundary with the same accuracy as the internal scheme, which should be complicated to implement. Based on such a reason, there is a great interest in the higher-order finite difference schemes. Since the 1950s, the compact finite difference schemes have been applied to solve partial differential equations more and more frequently, and more recently, the compact finite difference schemes have been extended to DPDEs, for instance, see [43] by proposing a compact multisplitting scheme for the nonlinear delay convection-reaction-diffusion equations, and [36] by applying the compact difference scheme to delay reaction-diffusion equations based on Crank-Nicolson scheme in temporal direction, and [42] by employing the compact difference scheme combined with extrapolation techniques to solve a class of neutral delay parabolic differential equations. The main