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THE IMMERSED FINITE VOLUME ELEMENT METHOD FOR SOME INTERFACE PROBLEMS WITH NONHOMOGENEOUS JUMP CONDITIONS

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Abstract. In this paper, an immersed finite volume element (IFVE) method is developed for solving some interface problems with nonhomogeneous jump conditions. Using the source removal technique of nonhomogeneous jump conditions, the new IFVE method is the finite volume element method applied to the equivalent interface problems with homogeneous jump conditions and have properties of the usual finite volume element method. The resulting IFVE scheme is simple and second order accurate with a uniform rectangular partition and the dual meshes. Error analyses show that the new IFVE method with usual $O(h^2)$ convergence in the L^2 norm and $O(h)$ in the H^1 norm. Numerical examples are also presented to demonstrate the efficiency of the new method.

Key words. Elliptic interface problem, non-homogeneous jump conditions, immersed finite volume element.

1. Introduction

Interface problems are often encountered in many important physical and industrial applications [11, 17].

In this paper, we consider the Poisson equation in a bounded Ω with an interface Γ in the domain,

$$-\Delta u = f \quad (x, y) \in \Omega \backslash \Gamma \subset R^2, \tag{1}$$

with a Dirichlet boundary condition

$$u(x, y) = g(x, y) \quad (x, y) \in \partial\Omega, \tag{2}$$

and jump conditions

$$[u]_\Gamma = w, \tag{3}$$

$$[u_n]_\Gamma = Q, \tag{4}$$

where

$$[u]_\Gamma = \lim_{\substack{(x,y)\to\Gamma \\ (x,y)\in\Omega^+}} u(x, y) - \lim_{\substack{(x,y)\to\Gamma \\ (x,y)\in\Omega^-}} u(x, y), \quad [u_n]_\Gamma = \lim_{\substack{(x,y)\to\Gamma \\ (x,y)\in\Omega^+}} u_n - \lim_{\substack{(x,y)\to\Gamma \\ (x,y)\in\Omega^-}} u_n,$$

with the notation $u_n = \nabla u \cdot n$. The interface $\Gamma \in C^2$ is a curve separating Ω into two subsets Ω^+ and Ω^-, and n is the unit outward normal vector of Γ pointing to the Ω^+ side, see Fig. 1 for an illustration.

In this paper, we assume that $w \neq 0$ and $Q \neq 0$. That is why we call such a problem that has nonhomogeneous jump conditions. The jump conditions can be obtained from the physics or mathematical derivations. For example, in Peskin's

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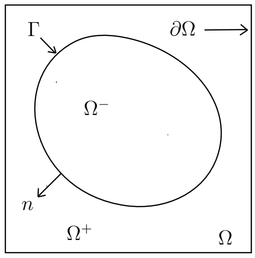

FIGURE 1. A diagram of the domain for the interface problem.

immersed boundary (IB) model [22], the pressure and its gradient are discontinuous, while the velocity is continuous, but the normal derivative of the velocity is discontinuous.

There are variety of methods that can be applied to solve the interface problem (1)-(4) numerically. First of all, in terms of the meshes, one can use a body fitted mesh or an unfitted mesh. With a body fitted mesh, the standard finite element method or finite volume element method is straightforward when $w = 0$. In this paper, we discuss a new finite volume element method using a uniform mesh with which there is almost no cost in the mesh generation; and we can utilize a fast Poisson solver, for example, the one from Fishpack [1]. Thus here we only give a brief literature review on numerical methods using a uniform unfitted mesh.

When $w = 0$, then the problem can be written as

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}) - \int_{\Gamma} Q(\mathbf{X}(s))\delta(\mathbf{x} - \mathbf{X}(s))ds, \quad \mathbf{x} \in \Omega,$$

$$u(\mathbf{x}) = g(\mathbf{x}), \ \mathbf{x} \in \partial\Omega,$$

where $\mathbf{X}(s) \in \Gamma$. In this case, we can use Peskin's IB method [22] to solve the interface problem. IB method is first order accurate and usually requires that the solution is continuous to have a convergent result. The immersed interface (II) method [12] is a second order accurate finite difference method even if $w \neq 0$ and $Q \neq 0$. To solve the resulting linear finite difference equations, both IB method and II method can call a fast Poisson solver. The main difference between the two methods are the right hand sides and the convergence rates. Related other methods include the matched interface and boundary (MIB) method [25], the ghost fluid (GF) method [3,19], the virtual node algorithm [2]. Another type of methods is based on finite element formulation. One type of approaches is to add some enrichment function near the interface [23] in which the number of degree of the freedom will be changed. Our method is more related to the immersed finite element (IFE) method [7,14,16] for which the structure and the number of degree of the freedom remain unchanged. IFE method has been applied to interface problems with nonhomogeneous jump conditions in [7,9,10,24]. In [7,15,18], the source removal technique was developed for treating nonhomogeneous jump conditions. In [10], a weak formulation inspired by the boundary condition capturing method [20] is proposed. In [24], the locally modified triangulations on irregular domains are proposed.

In this paper, we develop the *immersed finite volume element* (IFVE) method which is based on IFE method, similarly to [5,8]. The finite volume element method is also called generalized difference method [13]. Because the finite volume element