INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 13, Number 3, Pages 493–512

PARTIALLY OBSERVABLE STOCHASTIC OPTIMAL CONTROL

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Abstract. This paper is a survey on some recent results in optimal control and stochastic filtering. The goal is not to cover all recent developments in control and filtering, instead we focus on maximum principle for optimality of partial information backward or forward-backward stochastic differential equations and branching particle approximation of nonlinear filtering.

Key words. Branching particle system, forward-backward stochastic differential equation, numerical approximation, maximum principle, stochastic filtering.

1. Introduction

Stochastic control is the study of uncertain dynamical systems which can be controlled by decision makers so as to reach the best expected goals. In the real-world, the decision makers are usually only able to observe partially the state by other noisy observations. For example, in financial models, risky asset prices are observable but the appreciation rates of the assets are unavailable. See e.g. Xiong and Zhou [42] and the references therein. See also Huang, Wang and Wu [23] for optimal premium of insurance company with partial information. In these situations, we are facing optimal control problems of partially observable systems.

Such a kind of partially observed optimal control problem is composed of filtering and control. The filtering part is related to two stochastic processes: signal and observation. The signal process is what we want to estimate based on the observation which provides the information we can use. Analytical solutions to the filtering problems are rarely available in general. Thus, we have to resort to numerical schemes. Particle system approximation is an effective class of numerical schemes. The main idea is to represent the solution as a stochastic partial differential equation (SPDE) via a system of weighted particles whose locations and weights obey stochastic differential equations (SDEs) which can be solved numerically. The particle system approximation was studied in heuristic schemes by Gordon, Salmond and Ewing [20], Gordon, Salmond and Smith [21], Kitagawa [25], Carvalho et al. [3], Del Moral, Noyer and Salut [18]. Del Moral [14] considered a particle approximation for a model with independent observation noise that discounted past information. Florchinger and Gland [19] formulated a particle approximation for optimal filter. A rigorous proof of the convergent result for the particle filter is published by Del Moral [15], and independently, by Crisan and Lyons [11]. After that, many improvements were made by various authors. See e.g. Crisan and Lyons [10], Crisan [4], [5], [6], [7], Crisan, Gaines and Lyons [12], Crisan, Del Moral and Lyons [9], Crisan and Doucet [8], Del Moral and Guionnet [16], Del Moral and Miclo [17]. Later, Crisan and Xiong [13] proved a central limit type theorem for a new class of hybrid filters as well as for the original branching particle filters based on Kurtz and Xiong [26].

Received by the editors March 25, 2014 and, in revised form, October 18, 2015.
2000 *Mathematics Subject Classification.* 60H10, 60H35, 91B28, 93E11, 93E20.

In tradition, the partially observable optimal control problem is turned into a full information optimal control problem governed by Zakai equation, which is an SPDE driven by the observation process. However, this leads to an infinite dimensional optimal control problem, which is difficult to solve. See e.g. Bensoussan [2] for a systematic account. Recently, Wang and Wu [32] proposed a backward separation approach in order to study partially observed optimal control. The main idea is to decouple optimal control and state estimate by formally deducing optimal control first and then computing optimal filtering. An advantage of the approach is as follows. We use the original state and observation equation–which are finite dimensional–to calculate the variation, rather than the Zakai equation of the state based on the observation, which is infinite dimensional in general. Making use of this separating technique, lots of complicated stochastic calculus in infinite dimensional spaces are avoided. The approach is applicable to a broad class of control systems, say, backward or forward-backward stochastic differential equation (BSDE or FBSDE) systems. See e.g. Wang and Wu [33], Huang, Wang and Xiong [24], Wu [37], Shi and Wu [30], Xiao and Wang [39, 40], Xiao [38], Wang, Wu and Xiong [34, 35] for more details. See also Tang [31], Hu and Øksendal [22], Øksendal and Sulem [29], Meng [28], where optimal filtering was not studied.

The rest of this paper is organized as follows. The next section establishes several maximum principles for optimality of BSDEs and FBSDEs with partial information. To illustrate the maximum principles, a linear-quadratic (LQ) optimal control problem by means of BSDE is presented. Section 3 gives a brief introduction to the theory of nonlinear filter. A branching particle system is used to approximate the nonlinear filter in Section 4. Some numerical results will be presented in Section 5 to compare the particle filter, the optimal filter and the underlying state process. Finally, Section 6 lists some concluding remarks.

2. Maximum principle

Maximum principle is a set of necessary conditions satisfied by optimal solutions, which offers an approach for solving optimal control problems. This section is concerned with optimal control of BSDEs and FBSDEs with partial information. Two maximum principles for optimality are established, and an LQ example is used to shed light on the application of the maximum principles. These results are taken from the articles of Huang, Wang and Xiong [24], Wang, Wu and Xiong [34, 35].

2.1. The case of controlled BSDEs with partial information. We begin with a complete filtered probability space $(\Omega, \mathcal{F}^{W,Y}, (\mathcal{F}_t^{W,Y})_{0\le t\le 1}, \mathbb{P})$ on which an $\mathbb{R}^{m+d}$-valued standard Brownian motion (W, Y) is defined, and let $(\mathcal{F}_t^{W,Y})_{0\le t\le 1}$ be the natural filtration generated by (W, Y), and $\mathcal{F}^{W,Y} = \mathcal{F}_1^{W,Y}$. If $x : [0, 1] \times \Omega \to S$ is an $\mathcal{F}_t$-adapted and square-integrable process, we write $x \in L^2_{\mathcal{F}^{W,Y}}(0, 1; S)$; if $x : \Omega \to S$ is an $\mathcal{F}_1^{W,Y}$-measurable and square-integrable random variable, we write $x \in L^2_{\mathcal{F}_1^{W,Y}}(\Omega; S)$.

Let U be a non-empty convex subset of $\mathbb{R}^k$. Consider now a BSDE

$$\begin{cases} -dy_t = f(t, y_t, z_t, \bar{z}_t, v_t)dt - z_t dY_t - \bar{z}_t dW_t, \\ \quad y_1 = \xi, \end{cases} \tag{1}$$

where $\xi \in L^2_{\mathcal{F}_1^{W,Y}}(\Omega; \mathbb{R}^n)$, $v : [0, 1] \times \Omega \to U$ is a control process, and $f : [0, 1] \times \mathbb{R}^{n+n\times m+n\times d} \times U \to \mathbb{R}^n$ is a continuous mapping and satisfies