Analysis in Theory and Applications
Anal. Theory Appl., Vol. **30**, No. 3 (2014), pp. 290-295
DOI: 10.4208/ata.2014.v30.n3.5

On Some Inequalities Concerning Rate of Growth of Polynomials

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Received 25 June 2014; Accepted (in revised version) 19 July 2014

Available online 16 October 2014

Abstract. In this paper we consider a class of polynomials $P(z)=a_0+\sum_{v=t}^{n}a_v z^v$, $t\geq 1$, not vanishing in $|z|<k$, $k\geq 1$ and investigate the dependence of $\max_{|z|=1}|P(Rz)-P(rz)|$ on $\max_{|z|=1}|P(z)|$, where $1\leq r<R$. Our result generalizes and refines some known polynomial inequalities.

Key Words: Polynomial, zero, inequalitiy.

AMS Subject Classifications: 30A10, 30C10, 30D15

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1 Introduction

For an arbitrary entire function $f(z)$, let $M(f,r)=\max_{|z|=r}|f(z)|$ and $m(f,r)=\min_{|z|=r}|f(z)|$. For a polynomial $P(z)$ of degree n, it is known that

$$M(P,R)\leq R^n M(P,1),\quad R\geq 1. \tag{1.1}$$

The inequality (1.1) is a simple deduction from the Maximum Modulus Principle (see [7, pp. 442]). It was shown by Ankeny and Rivlin [1] that if $P(z)$ does not vanish in $|z|<1$, then (1.1) can be replaced by

$$M(P,R)\leq \Big(\frac{R^n+1}{2}\Big)M(P,1),\quad R\geq 1. \tag{1.2}$$

The bound in (1.2) was further improved by Aziz and Dawood [4], who under the same hypothesis proved

$$M(P,R)\leq \Big(\frac{R^n+1}{2}\Big)M(P,1)-\Big(\frac{R^n-1}{2}\Big)m(P,1),\quad R\geq 1. \tag{1.3}$$

Recently Mir, Dewan and Singh [2] investigated the dependence of $\max_{|z|=1}|P(Rz)-P(z)|$ on $M(P,1)$ and $m(P,K)$, where $R>1$ and proved the following result.

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Theorem 1.1. *Let* $P(z)=a_0+\sum_{v=t}^{n}a_v z^v$, $t\geq 1$, *be a polynomial of degree* n *not vanishing in* $|z|<k$, *where* $k\geq 1$ *then for every* $R>1$ *and* $|z|=1$,

$$|P(Rz)-P(z)|\leq\Big(\frac{R^n-1}{1+\psi_0(R)}\Big)\{M(P,1)-m(P,k)\}, \tag{1.4}$$

where

$$\psi_0(R)=k^{t+1}\left\{\frac{\left(\frac{R^t-1}{R^n-1}\right)\frac{|a_t|k^{t-1}}{|a_0|-m(P,k)}+1}{\left(\frac{R^t-1}{R^n-1}\right)\frac{|a_t|k^{t+1}}{|a_0|-m(P,k)}+1}\right\}. \tag{1.5}$$

From the inequality (1.4), *it follows that*

$$M(P,R)\leq\Big(\frac{R^n+\psi_0(R)}{1+\psi_0(R)}\Big)M(P,1)-\Big(\frac{R^n-1}{1+\psi_0(R)}\Big)m(P,k). \tag{1.6}$$

It is easy to verify (for example by the derivative test) that for every n *and* $R>1$, *the function*

$$\Big(\frac{R^n+x}{1+x}\Big)M(P,1)-\Big(\frac{R^n-1}{1+x}\Big)m(P,k),$$

a non-increasing in x. *If we combine this fact with* $\psi_0(R)\geq k^t$, *for* $t\geq 1$, *we can easily obtain from* (1.6) *that*

$$M(P,R)\leq\Big(\frac{R^n+k^t}{1+k^t}\Big)M(P,1)-\Big(\frac{R^n-1}{1+k^t}\Big)m(P,k), \tag{1.7}$$

which is clearly a generalization of (1.3).

It is worth to mention that Theorem 1.1 was also independently proved by Aziz and Aliya [3]. In the same paper Aziz and Aliya [3] proved the following more general result containing Theorem 1.1 as a special case.

Theorem 1.2. *If* $P(z)=a_0+\sum_{v=t}^{n}a_v z^v$, $t\geq 1$, *is a polynomial of degree* n *having no zeros in* $|z|<k$, $k\geq 1$, *then for every* $R>r\geq 1$, $0\leq\lambda\leq 1$ *and* $|z|=1$,

$$|P(Rz)-P(rz)|\leq\Big(\frac{R^n-r^n}{1+\psi_1(R)}\Big)\{M(P,1)-\lambda m(P,k)\}, \tag{1.8}$$

where

$$\psi_1(R)=k^{t+1}\left\{\frac{\left(\dfrac{R^t-r^t}{R^n-r^n}\right)\frac{|a_t|k^{t-1}}{|a_0|-\lambda m(P,k)}+1}{\left(\dfrac{R^t-r^t}{R^n-r^n}\right)\frac{|a_t|k^{t+1}}{|a_0|-\lambda m(P,k)}+1}\right\}. \tag{1.9}$$

For $r=\lambda=1$, *the inequality* (1.8) *reduces to* (1.4).