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NON-ORTHOGONAL P-WAVELET PACKETS ON THE HALF-LINE

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Abstract. In this paper, the notion of p-wavelet packets on the positive half-line $\mathbb{R}^+$ is introduced. A new method for constructing non-orthogonal wavelet packets related to Walsh functions is developed by splitting the wavelet subspaces directly instead of using the low-pass and high-pass filters associated with the multiresolution analysis as used in the classical theory of wavelet packets. Further, the method overcomes the difficulty of constructing non-orthogonal wavelet packets of the dilation factor $p > 2$.

Key words: *p-Multiresolution analysis, p-wavelet packets, Riesz basis, Walsh function, Walsh-Fourier transform*

AMS (2010) subject classification: 42C40, 42C10, 42C15

1 Introduction

In the early nineties a general scheme for the construction of wavelets was defined. This scheme is based on the notion of multiresolution analysis (MRA) introduced by Mallat[16]. Immediately specialists started to implement new wavelet systems and in recent years, the concept MRA of $\mathbf{R}^n$ has been extended to many different setups, for example, Dahlke introduced multiresolution analysis and wavelets on locally compact Abelian groups[5], Lang[14] constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group $\mathcal{C}$ by following the procedure of Daubechies [6] via scaling filters and these wavelets turn out to be certain lacunary Walsh series on the real line. On the otherhand, Jiang et al.[13] pointed out a method for constructing orthogonal wavelets on local field $\mathbf{K}$ with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(\mathbf{K})$. Subsequently, the tight wavelet frames on local fields were constructed by Li and Jiang in [15]. Farkov[7] extended the results of Lang[14] on the wavelet analysis on the Cantor dyadic group $\mathcal{C}$ to the locally compact Abelian group G_p which is defined for an integer $p \geq 2$ and coincides with $\mathcal{C}$ when $p = 2$. Concerning the construction of wavelets on a half-line, Farkov [8] has given the general construction of all compactly supported orthogonal p-wavelets in $L^2(\mathbf{R}^+)$ and proved necessary and sufficient conditions for scaling filters with p^n many terms $(p,n \geq 2)$ to generate a p-MRA analysis in $L^2(\mathbf{R}^+)$. These studies were continued by Farkov and his colleagues in [9,10], where they have given some new algorithms for constructing the corresponding biorthogonal and nonstationary wavelets related to the Walsh functions on the positive half-line $\mathbf{R}^+$. On the otherhand, Shah and Debnath[21] have constructed dyadic wavelet frames on the positive half-line $\mathbf{R}^+$ using the Walsh-Fourier transform and have established a necessary condition and a sufficient condition for the system $\left\{2^{j/2}\psi(2^j x \ominus k) : j \in \mathbf{Z}, k \in \mathbf{Z}^+\right\}$ to be a frame for $L^2(\mathbf{R}^+)$.

It is well-known that the classical orthonormal wavelet bases have poor frequency localization. For example, if the wavelet ψ is band limited, then the measure of the supp of $(\psi_{j,k})^\wedge$ is 2^j-times that of supp $\hat{\psi}$. To overcome this disadvantage, Coifman et al.[4] constructed univariate orthogonal wavelet packets. The fundamental idea of wavelet packet analysis is to construct a library of orthonormal bases for $L^2(\mathbf{R})$, which can be searched in real time for the best expansion with respect to a given application.

Let $\varphi(x)$ and $\psi(x)$ be the scaling function and the wavelet function associated with a multiresolution analysis $\left\{V_j\right\}_{j\in\mathbf{Z}}$. Let W_j be the corresponding wavelet subspaces:

$$W_j = \overline{\text{span}}\left\{\psi_{j,k} : k \in \mathbf{Z}\right\}.$$

Using the low-pass and high-pass filters associated with the MRA, the space W_j can be split into two orthogonal subspaces, each of them can further be split into two parts. Repeating this process j times, W_j is decomposed into 2^j subspaces each generated by integer translates of a single function. If we apply this to each W_j, then the resulting basis of $L^2(\mathbf{R})$ which will consist of integer translates of a countable number of functions, will give a better frequency localization. This basis is called the *wavelet packet basis*. To describe this more formally, we introduce a parameter n to denote the frequency. Set $\omega_0 = \varphi$ and define recursively

$$\omega_{2n}(x) = \sum_{k\in\mathbf{Z}} h_k \omega_n(2x-k), \qquad \omega_{2n+1}(x) = \sum_{k\in\mathbf{Z}} g_k \omega_n(2x-k),$$

where $\{h_k\}_{k\in\mathbf{Z}}$ and $\{g_k\}_{k\in\mathbf{Z}}$ are the low-pass filter and high-pass filter corresponding to $\varphi(x)$ and $\psi(x)$, respectively. Chui and Li[2] generalized the concept of orthogonal wavelet packets to