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Some L^γ Inequalities for the Polar Derivative of a Polynomial

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Abstract. In this paper, we consider an operator D_α which maps a polynomial $P(z)$ in to $D_\alpha P(z) := np(z) + (\alpha - z)P'(z)$, where $\alpha \in \mathbb{C}$ and obtain some L^γ inequalities for lucanary polynomials having zeros in $|z| \le k \le 1$. Our results yields several generalizations and refinements of many known results and also provide an alternative proof of a result due to Dewan et al. [7], which is independent of Laguerre's theorem.

Key Words: Polar derivative, polynomials, L^γ-inequalities in the complex domain, Laguerre's theorem.

AMS Subject Classifications: 30A10, 30C10, 30C15

1 Introduction

Let P_n be the class of polynomials

$$P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$$

of degree n. For $P \in P_n$, define

$$\|P\|_\gamma := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^\gamma \right\}^{\frac{1}{\gamma}}, \quad \gamma > 0,$$

$$\|P\|_\infty := \max_{|z|=1} |P(z)|, \quad m := \min_{|z|=k} |P(z)| \quad \text{and} \quad m_1 := \min_{|z|=1} |P(z)|.$$

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For fixed μ, $1\le\mu\le n$, let $P_{n,\mu}$, denote the class of polynomials

$$P(z)=a_nz^n+\sum_{v=\mu}^{n}a_{n-v}z^{n-v}$$

of degree n having all zeros in $|z|\le k$, $k\le 1$.

If $P\in P_n$, then according to the following well-known Bernstein's inequality (for reference see [5]), we have

$$\|P'\|_\infty\le n\|P\|_\infty. \tag{1.1}$$

Equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin.

For the class of polynomials $P\in P_n$ having all zeros in $|z|\le 1$, Turán [14] proved that

$$\|P'\|_\infty\ge\frac{n}{2}\|P\|_\infty. \tag{1.2}$$

Inequality (1.2) was refined by Aziz and Dawood [1] and they proved under the same hypothesis that

$$\|P'\|_\infty\ge\frac{n}{2}\Big\{\|P\|_\infty+m_1\Big\}. \tag{1.3}$$

Both the inequalities (1.2) and (1.3) are best possible and become equality for polynomials $P(z)=\alpha z^n+\beta$, where $|\alpha|=|\beta|$. As an extension of (1.2), it was shown by Malik [12], that if $P\in P_{n,1}$, then

$$\|P'\|_\infty\ge\frac{n}{1+k}\|P\|_\infty, \tag{1.4}$$

where as the corresponding extension of (1.3) and a refinement of (1.4) was given by Govil [9] who under the same hypothesis proved that

$$\|P'\|_\infty\ge\frac{n}{1+k}\Big\{\|P\|_\infty+\frac{m}{k^{n-1}}\Big\}. \tag{1.5}$$

In the literature, there already exist some refinements and generalizations of all the above inequalities, for example see Aziz and Shah [4], Dewan, Mir and Yadav [8], Govil, Rahman and Schemeisser [10], Dewan, Singh and Lal [6], etc.

Aziz and Shah [4] (see also Dewan, Mir and Yadav [8]) generalized inequality (1.5) and proved that, if $P\in P_{n,\mu}$, then

$$\|P'\|_\infty\ge\frac{n}{1+k^\mu}\Big\{\|P\|_\infty+\frac{m}{k^{n-\mu}}\Big\}. \tag{1.6}$$

For $\mu=1$, inequality (1.6) reduces to inequality (1.5).

For a complex number α and for $P\in P_n$, let

$$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z).$$