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CONVERGENCE OF A CELL-CENTERED FINITE VOLUME METHOD AND APPLICATION TO ELLIPTIC EQUATIONS

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Abstract. We study the consistency and convergence of the cell-centered Finite Volume (FV) external approximation of $H_0^1(\Omega)$, where a 2D polygonal domain Ω is discretized by a mesh of convex quadrilaterals. The discrete FV derivatives are defined by using the so-called Taylor Series Expansion Scheme (TSES). By introducing the Finite Difference (FD) space associated with the FV space, and comparing the FV and FD spaces, we prove the convergence of the FV external approximation by using the consistency and convergence of the FD method. As an application, we construct the discrete FV approximation of some typical elliptic equations, and show the convergence of the discrete FV approximations to the exact solutions.

Key words. Finite Volume method, Taylor Series Expansion Scheme (TSES), convergence and stability, convex quadrilateral meshes.

1. Introduction

In engineering, fluid dynamics, and physics more recently, the Finite Volume (FV) discretization method is widely used because of its local conservation property of the flux on each control volume. From the numerical analysis point of view, many different types of FV methods, depending on the way of computing the discrete fluxes, have been introduced and analyzed up to this day. Concerning the varieties of the FV methods and their applications, we refer the readers to, e.g., [52, 32, 51, 18, 24] for general references, and to [36, 1, 37, 47, 58, 60, 48, 38, 40, 49, 11] for the computational applications.

In proving the convergence of the cell-centered FV method, one specific difficulty is due to the *weak consistency* of the FV method. Namely, the companion discrete FV derivative arising in the discrete integration by parts does not usually converge strongly to the corresponding derivative of the limit function. To overcome this technical difficulty, in an important earlier work, the authors of [32] employed a discrete compactness argument for the FV space, even for linear problems. Since then, using this approach, further analysis of the cell-centered FV method has been made in, e.g, [28, 27, 12, 31, 41, 8, 14]. A different approach was introduced in our earlier works [39, 42] to prove the convergence of the cell-centered FV method. More precisely, we introduced there the Finite Difference (FD) space which is associated with the FV space, and compared the FV and FD spaces by defining a map between them. Then, thanks to the consistency and convergence of the FD method which are proven in a classical way, the convergence of the FV method is inferred. This approach was conducted in [39, 42] for the study of the cell-centered FV method when the domain considered has a rectangular mesh, whereas more general meshes are desirable for FV which are specifically aimed at handling complicated geometries. For a different type of FV methods other than the cell-centered FV, the convergence of, e.g., the cell-vertex FV method is well-studied in, e.g., [54, 55, 56, 10, 53, 59].

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There is a broad class of FV methods which correspond to various equations and applications, and to various strategies of approximation. After choosing, for a given mesh, the nodal points and the reconstructed functions, one needs to define an approximation of the derivatives (which is not easy on a general mesh). For elliptic problems which admit a weak (variational) formulation, the gradient schemes (studied in, e.g., [34, 26]) consist in mimicking the variational formulation by replacing the exact derivatives by the approximate derivatives. Another more general direction applying to all classes of equations and conservation laws, consists in integrating the conservation law on the control volume and then looking for approximations of the fluxes. Our approach relates to the gradient schemes. Using cell-centered unknowns, we approach the spatial derivatives using the so-called Taylor Series Expansion Scheme (TSES) method that was introduced in the early 90's in the engineering literature, see, e.g., [46, 52, 45, 18]. Note that the TSES is commonly considered in the engineering fluid mechanics, whereas the MPFA approach, [9, 3], not studied here is considered in the petroleum and hydrogeology literatures. We can then define approximate variational problems and study the convergence of the approximate solutions to those of the exact ones. The construction of the TSES method is closely related to, e.g., that of the so-called *diamond scheme* in [22] or the Discrete Duality Finite Volume scheme in [44, 23] according to this terminology which was subsequently introduced; see Remark 3.2 below. See other related works as well in, e.g., [43, 30, 16, 2].

As we said, there is a substantial body of work related to the numerical analysis of the FV methods; see, e.g., the review articles [32], and more recently [34, 26]; see also, e.g., [31, 41] and the references quoted in these articles. Despite their importance and major interest, the existing works deal with objectives different than ours and do not cover our objectives. These works are generally motivated by reservoir (underground) flows and address the corresponding equations, and, on the mathematical side, they use compactness arguments to prove convergence, even for linear equations. Due to the growing importance of FV methods, and the considerable difficulties for proving their convergence, it is clear that there will be many more works in years to come on the numerical analysis of FV methods, and there is need to diversify the available tools. This article, like earlier works [39, 42], is generally motivated by classical or geophysical fluid mechanics; it uses a form of the FV method, the TSES method, which is not dealt with in the review articles previously mentioned; and it uses the comparison with a related Finite Difference method instead of compactness arguments. Another major difference between prior works and this article is that, in, e.g., [32, 34, 26], the FV method and its analysis is taylored to one specific equation in divergence form and, as far as we understand, the work needs to be redone or suitably adapted if we consider a different equation with, e.g., lower order terms as in equation (124) in this article. On the contrary, our approach consists in approximating the underlying function space of type $H^1(\Omega)$, leaving all flexibility for the equations whose coefficients can be nonhomogeneous and nonisotropic. Finally it is noteworthy that [24] emphasizes the use of the maximum principle which is mostly not relevant to classical or geophysical fluid mechanics (nor to multi-species underground reservoir flows which produce systems).

In this article, to prove the consistency and convergence of the TSES Finite Volume approximation of $H_0^1(\Omega)$, (which is equivalent to verifying the properties $(\mathcal{C}1)$ and $(\mathcal{C}2)$ below), we impose some conditions $(\mathcal{H}1)$-$(\mathcal{H}5)$ on the mesh. $(\mathcal{H}1)$-$(\mathcal{H}3)$ are standard hypotheses which guarantee that the mesh is not too distorted.