INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 12, Number 4, Pages 636–663

A T-ψ FINITE ELEMENT METHOD FOR A NONLINEAR DEGENERATE EDDY CURRENT MODEL WITH FERROMAGNETIC MATERIALS

TONG KANG AND TAO CHEN

(Communicated by Xue-cheng Tai)

Abstract. This paper is devoted to the study of a fully discrete $\boldsymbol{T}$-ψ finite element method based on $\boldsymbol{H}$-decomposition to solve a nonlinear degenerate transient eddy current problem with ferromagnetic materials. Here, the ferromagnetic properties are linked by a power material law. We first design a nonlinear time-discrete scheme for approximation in suitable function spaces. We show the well-posedness of the semidiscrete problem and prove the convergence of the nonlinear scheme by the Minty-Browder technique. Finally, we suggest a fully discrete scheme, derive its error estimate and give some numerical experiments to validate the theoretical result.

Key words. nonlinear degenerate eddy current problem, $\boldsymbol{T}$-ψ method, nodal elements, convergence, and error estimates.

1. Introduction

The growing industrial applications of superconducting materials increase the necessity for accurate numerical methods and their solid mathematical analysis. To derive a precise mathematical model, we usually use the eddy current approximation of Maxwell's equations by formally dropping the displacement currents:

$$
\begin{cases} \dfrac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0, \\ \nabla \times \boldsymbol{H} = \boldsymbol{J}, \end{cases} \tag{1}
$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ stands for the magnetic induction, $\boldsymbol{H}$ denotes the magnetic field and $\boldsymbol{J}$ is the current density.

Let $\Omega \subset \mathbb{R}^3$ be a sufficiently large, bounded polyhedron with the connected boundary $\partial\Omega$. This domain consists of some simply-connected convex subdomains occupied by ferromagnetic materials, which are denoted by Ω_c with the boundary $\partial\Omega_c$. Let the complement $\Omega_e = \Omega \backslash \overline{\Omega}_c$ be the nonconducting domain. Taking into account Ohm's law,

$$
\boldsymbol{J} = \sigma \boldsymbol{E},
$$

where σ is the conductivity. We assume that σ is piecewise constants in Ω_c and vanishes outside Ω_c, and there exist two constants $\sigma_{\min}$ and $\sigma_{\max}$ such that $0 < \sigma_{\min} \leq \sigma \leq \sigma_{\max}$ in Ω_c. The time-dependent magnetic variables are related as follows:

$$
\boldsymbol{B}(\boldsymbol{H}) = \begin{cases} \mu_0\big(\boldsymbol{H} + \boldsymbol{M}(\boldsymbol{H})\big) & \text{in } \Omega_c, \\ \mu_0 \boldsymbol{H} & \text{in } \Omega_e, \end{cases} \tag{2}
$$

Received by the editors January 10, 2014 and, in revised form, January 10, 2015.
2000 *Mathematics Subject Classification.* 35Q60, 65N30, 65M60.
This work was supported by R&D of Key Instruments and Technologies for Deep Resources Prospecting (the National R&D Projects for Key Scientific Instruments) under grant number ZDYZ2012-1-02-04 and China NSF under grant 91130015.
Corresponding author: chentao@cuc.edu.cn.

where μ_0 denotes the magnetic permeability of free space and $\boldsymbol{M}$ stands for the magnetization vector. One can characterize the relationship between $\boldsymbol{M}$ and $\boldsymbol{H}$ by a material law (for some $0 < \alpha < 1$)

$$\boldsymbol{M}(\boldsymbol{H}) = \begin{cases} |\boldsymbol{H}|^{\alpha-1}\boldsymbol{H}, & \text{if } |\boldsymbol{H}| \le 1, \\ |\boldsymbol{H}|^{-1}\boldsymbol{H}, & \text{if } |\boldsymbol{H}| > 1. \end{cases} \tag{3}$$

We consider the eddy current problem (1) with magnetic and anisotropic materials. Assume the boundary condition

$$\boldsymbol{H} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \times (0, T]$$

and the initial condition

$$\boldsymbol{H}(\boldsymbol{x}, 0) = \boldsymbol{H}_0(\boldsymbol{x}),$$

where $\boldsymbol{n}$ is regarded as the unit outward normal vector on $\partial\Omega$ or $\partial\Omega_c$. For physical reasons, it is supposed that $\nabla \cdot \boldsymbol{B}(\boldsymbol{H}_0) = 0$. Then we obtain the following initial boundary value problem:

$$\begin{cases} \dfrac{\partial}{\partial t}\boldsymbol{B}(\boldsymbol{H}) + \nabla \times \left(\dfrac{1}{\sigma}\nabla \times \boldsymbol{H}\right) = \boldsymbol{0} & \text{in } \Omega_c, \\ \nabla \cdot \dfrac{\partial}{\partial t}(\mu_0 \boldsymbol{H}) = 0 & \text{in } \Omega_e, \\ \boldsymbol{H} \cdot \boldsymbol{n} = 0 & \text{on } \partial\Omega, \\ \boldsymbol{H}(\cdot, 0) = \boldsymbol{H}_0(\cdot) & \text{in } \Omega. \end{cases} \tag{4}$$

The nonlinear PDEs of the type (4) have some applications in superconductors (see [13, 14]). It is well known that high-field (hard) type-II superconductors are not ideal conductors of electric current and are usually treated as electrically nonlinear conductors. The process of electromagnetic field penetration in such devices is the process of nonlinear diffusion. The equations describing the process can degenerate. For an overview of models with some hierarchy structure we refer the readers to [4, 10]. The magnetization of type-II superconductors in a nonstationary external magnetic field can also be formulated in terms of a scalar p-Laplacian equation if the magnetic field lies only in one direction. This situation has been studied in many papers, e.g. [3, 29, 30]. Slodička applied the backward Euler scheme to this type of equations for discretization in time and derived the error estimates for a degenerate problem in [24] and an application in superconductors in [25]. These error estimates for the time-discretization in both papers are suboptimal. Some similar works can be also found in [8, 15, 19, 26].

To solve quasistationary Maxwell's equations by the finite element methods, various formulations different in the choices of the primary unknowns are suggested, such as, direct approaches based on the electric/magnetic field, and indirect approaches based on potential fields (e.g. the $\boldsymbol{A}$-ϕ method from $\boldsymbol{E}$-decomposition and the $\boldsymbol{T}$-ψ method from $\boldsymbol{H}$-decomposition). The main difficulty in application of nodal elements for direct approaches is that the normal component of the field is discontinuous on the interface between different materials due to the presence of inhomogeneous mediums, but indirect approaches can avoid it.

The $\boldsymbol{T}$-ψ method is to decompose the magnetic field into summation of a vector potential $\boldsymbol{T}$ and the gradient of a scalar potential ψ in the simply-connected conductors, and only the gradient of a scalar potential ψ outside the conductors [1, 2, 6, 16, 17, 21, 33], afterward to approximate both potential fields by piecewise polynomial functions. The $\boldsymbol{T}$-ψ method has some advantages: First, although introducing the vector and scalar potentials increases the number of unknowns and