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Asymptotic Behavior for a Viscoelastic Wave Equation with a Time-varying Delay Term

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Abstract. The following viscoelastic wave equation with a time-varying delay term in internal feedback

$$|u_t|^\rho u_{tt}-\Delta u-\Delta u_{tt}+\int_0^t g(t-s)\Delta u(s)\mathrm{d}s+\mu_1 u_t(x,t)+\mu_2 u_t(x,t-\tau(t))=0,$$

is considered in a bounded domain. Under appropriate conditions on μ_1, μ_2 and on the kernel g, we establish the general decay result for the energy by suitable Lyapunov functionals.

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1 Introduction

In this paper, we consider the initial boundary value problem for a nonlinear viscoelastic equation with a linear damping and a time-varying delay term of the form:

$$\begin{aligned}
&|u_t|^\rho u_{tt}-\Delta u-\Delta u_{tt}+\int_0^t g(t-s)\Delta u(s)\mathrm{d}s+\mu_1 u_t(x,t)\\
&\qquad +\mu_2 u_t(x,t-\tau(t))=0, \qquad \text{in } \Omega\times(0,\infty), && (1.1)\\
&u_t(x,t)=f_0(x,t), \qquad x\in\Omega,\ t\in[-\tau(0),0), && (1.2)\\
&u(x,0)=u_0(x), u_t(x,0)=u_1(x), \qquad x\in\Omega, && (1.3)\\
&u(x,t)=0, \qquad x\in\partial\Omega, t\geq 0, && (1.4)
\end{aligned}$$

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where $\rho>0$, $\Omega\subset R^N$ $(N\geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$. Moreover, μ_1 and μ_2 are real constants with $\mu_1>0$, $\tau(t)>0$ represents the time-varying delay, g is the kernel of the memory term and the initial data (u_0,u_1,f_0) are given functions belonging to suitable spaces.

It is well known that delay effects, which arise in many practical problems, might induce some instabilities, see $[1-6]$. Hence, questions related to the behavior of solutions for the PDEs with time delay effects have become active area of research in recent years. Many authors have focused on this problem and several results concerning existence, decay and instability have been obtained, see $[2-12]$ and reference therein. In this regard, Datko et al. [4] showed that a small delay in a boundary control is a source of instability. Nicaise et al. [7] studied a system of wave equation with a linear boundary damping term with a delay as follows

$$\begin{cases} u_{tt}-\Delta u=0, & \text{in } \Omega\times(0,\infty),\\ u(x,t)=0, & x\in\Gamma_0, t\geq 0,\\ \frac{\partial}{\partial\nu}(x,t)=\mu_1 u_t(x,t)+\mu_2 u_t(x,t-\tau), & \text{in } \Gamma_1\times(0,\infty),\\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), & x\in\Omega,\\ u_t(x,t-\tau)=f_0(x,t-\tau), & x\in\Omega,\ t\in(0,\tau). \end{cases} \tag{1.5}$$

where ν is the unit outward normal to $\partial\Omega$. Under the condition

$$\mu_2<\mu_1, \tag{1.6}$$

they established a stabilization result by applying inequalities obtained from Carleman estimates for the wave equation by Lasiecka et al. [13] and by using compactness-uniqueness arguments. Conversely, if (1.6) does not hold, they showed that there exists a sequence of delays for which the corresponding solution of (1.5) is unstable. And, they also obtained the same results if both the damping and the delay act in the domain.

The case of time-varying delay in the wave equation has been studied by Nicaise et al. [10] in one space dimension, in which they obtained an exponential decay result subject to the condition

$$\mu_2\leq\sqrt{1-d}\,\mu_1, \tag{1.7}$$

where d is a constant such that

$$\tau'(t)\leq d<1, \qquad \forall t>0. \tag{1.8}$$

Later, under the condition $|\mu_2|<\sqrt{1-d}\,\mu_1$ in which the positivity of the coefficient μ_2 is not necessary, Nicaise et al. [11] extended this result to general space dimension. In fact, they proved exponential stability of the solution for the wave equation with a time-varying delay in the boundary condition in a bounded and smooth domain in R^N. Recently, inspired the works of Nicaise et al. [11] and M. Kirane et al. [5], Liu [14] considered