INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 12, Number 4, Pages 750–777



# AN INTERIOR PENALTY DISCONTINUOUS GALERKIN METHOD FOR A CLASS OF MONOTONE QUASILINEAR ELLIPTIC PROBLEMS

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**Abstract.** A family of interior penalty $hp$-discontinuous Galerkin methods is developed and analyzed for the numerical solution of the quasilinear elliptic equation $-\nabla \cdot (\mathbf{A}(\nabla u)\nabla u) = f$ posed on the open bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$. Subject to the assumption that the map $\mathbf{v} \mapsto \mathbf{A}(\mathbf{v})\mathbf{v}$, $\mathbf{v} \in \mathbb{R}^d$, is Lipschitz continuous and strongly monotone, it is proved that the proposed method is well-posed. *A priori* error estimates are presented of the error in the broken $H^1(\Omega)$-norm, exhibiting precisely the same $h$-optimal and mildly $p$-suboptimal convergence rates as obtained for the interior penalty approximation of linear elliptic problems. *A priori* estimates for linear functionals of the error and the $L^2(\Omega)$-norm of the error are also established and shown to be $h$-optimal for a particular member of the proposed family of methods. The analysis is completed under fairly weak conditions on the approximation space, allowing for non-affine and curved elements with multilevel hanging nodes. The theoretical results are verified by numerical experiments.

**Key words.** $hp$-discontinuous Galerkin methods, interior penalty methods, second-order quasilinear elliptic problems.

## 1. Introduction

Over the past two decades, discontinuous Galerkin (DG) finite element methods have emerged as an effective and popular choice for the numerical solution of a wide range of partial differential equations. This is mainly stimulated by their high degree of locality, their extreme flexibility with respect to $hp$-adaptive mesh refinement, and their natural ability to accommodate high-order discretizations for hyperbolic problems in a locally conservative manner without excessive numerical stabilization. As it stands, there exists a vast amount of literature on the *a priori* error analysis of DG methods for linear problems; we refer to the recent book of Di Pietro & Ern [9] for a comprehensive overview of the most prominent results. For nonlinear problems, however, there are still relatively few results available; we mention the works of Houston *et al.* [18], Ortner & Süli [23], Gudi & Pani [16], Gudi *et al.* [14, 15], Dolejší *et al.* [10, 11], Bustinza & Gatica [5], Bi & Lin [4], and Congreve *et al.* [8]. It is fair to say that the extension of DG methods from linear to nonlinear problems is non-obvious in many cases, particularly with respect to the proper formulation of the element boundary terms, and that the analysis turns out to be more challenging.

In this article, we present and analyze a family of interior penalty DG methods for the numerical solution of the following class of quasilinear elliptic boundary value problems. Let $\Omega$ be an open bounded domain in $\mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary $\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$, where $\Gamma_{\mathrm{D}} \neq \emptyset$ and $\Gamma_{\mathrm{N}} = \partial\Omega \setminus \Gamma_{\mathrm{D}}$. Denoting by $\mathbf{n}\colon \Gamma_{\mathrm{N}} \to \mathbb{R}^d$ the unit outward normal to $\Gamma_{\mathrm{N}}$, our model problem of interest is stated as follows: find

Received by the editors March 13, 2014, accepted on May 29, 2015.
2000 *Mathematics Subject Classification.* 65N12, 65N15, 65N30.

$u \colon \bar{\Omega} \to \mathbb{R}$ such that

$$-\nabla \cdot (\mathbf{A}(\mathbf{x}, \nabla u)\nabla u) = f \qquad \text{in } \Omega, \tag{1a}$$

$$u = g_{\mathrm{D}} \qquad \text{on } \Gamma_{\mathrm{D}}, \tag{1b}$$

$$\mathbf{A}(\mathbf{x}, \nabla u)\nabla u \cdot \mathbf{n} = g_{\mathrm{N}} \qquad \text{on } \Gamma_{\mathrm{N}}, \tag{1c}$$

where $\mathbf{A} \in [L^\infty(\bar{\Omega} \times \mathbb{R}^d)]^{d,d}$, $f \in L^2(\Omega)$, $g_{\mathrm{D}} \in H^{1/2}(\Gamma_{\mathrm{D}})$ and $g_{\mathrm{N}} \in L^2(\Gamma_{\mathrm{N}})$. In what follows, we assume that, for $\mathbf{x} \in \bar{\Omega}$ and $\mathbf{v} \in \mathbb{R}^d$, the nonlinear map $\mathbf{v} \mapsto \mathbf{A}(\mathbf{x}, \mathbf{v})\mathbf{v}$ is *Lipschitz continuous* and *strongly monotone*, as phrased by the following statement.

**Assumption 1.1.** *There exist constants $C_{\mathbf{A}} \geq M_{\mathbf{A}} > 0$ such that, for all $\mathbf{x} \in \bar{\Omega}$ and all $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^d$,*

$$|\mathbf{A}(\mathbf{x}, \mathbf{v}_1)\mathbf{v}_1 - \mathbf{A}(\mathbf{x}, \mathbf{v}_2)\mathbf{v}_2| \leq C_{\mathbf{A}} \, |\mathbf{v}_1 - \mathbf{v}_2|, \tag{2}$$

$$(\mathbf{A}(\mathbf{x}, \mathbf{v}_1)\mathbf{v}_1 - \mathbf{A}(\mathbf{x}, \mathbf{v}_2)\mathbf{v}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2) \geq M_{\mathbf{A}} \, |\mathbf{v}_1 - \mathbf{v}_2|^2. \tag{3}$$

Subject to the above assumption, one can show that problem (1) admits a unique weak solution $u \in H^1(\Omega)$. In passing, we note that problems of the type (1) satisfying Assumption 1.1 arise in several applications. A classic example is mean curvature flow, for which $\mathbf{A}(\mathbf{x}, \nabla u) = (1 + |\nabla u|^2)^{-1/2} \, \mathbf{I}$ with $\mathbf{I}$ the $d \times d$ identity matrix; this has applications in image processing and interface modeling in two-fluid flows, among others. Another example is the modeling of non-Newtonian fluids. For the sake of notational simplicity, we henceforth suppress the dependence of $\mathbf{A}(\mathbf{x}, \mathbf{v})$ on $\mathbf{x}$ and simply write $\mathbf{A}(\mathbf{v})$ instead.

The development of DG methods for problems of the type (1) has also been pursued by several other researchers. In [5], an $h$-version local DG method is developed and analyzed exhibiting optimal error estimates in the broken $H^1(\Omega)$-norm and $L^2(\Omega)$-norm. The development and analysis of $hp$-version interior penalty DG methods is initiated by Houston *et al.* [18]. Quasi-optimal error estimates are presented for the error in the broken $H^1(\Omega)$-norm, which are optimal in the mesh size $h$ and mildly suboptimal in the polynomial degree $p$, by half an order in $p$. Estimates for the error in the $L^2(\Omega)$-norm are not presented, but numerical experiments reveal the convergence in the $L^2(\Omega)$-norm to be suboptimal. This suboptimality is caused by so-called dual inconsistency of the method due to a particular formulation of the element boundary terms. Difficulties with respect to the proper formulation of the element boundary terms have motivated other researchers to consider the development of *incomplete* interior penalty DG methods; cf. [4, 10, 23]. In [15], a family of interior penalty DG methods is presented and analyzed with a particular choice of the element boundary terms, for which quasi-optimal $hp$-error estimates are derived in both the broken $H^1(\Omega)$-norm and $L^2(\Omega)$-norm.

The purpose of this article is to present and analyze a new family of interior penalty $hp$-DG methods for the numerical solution of (1) with quasi-optimal $hp$-error estimates in both the broken $H^1(\Omega)$-norm and $L^2(\Omega)$-norm. As in [18] and [15], our family of methods depends on the parameter $\theta \in [-1, 1]$. In the linear setting of $\mathbf{A}(\cdot) = \mathbf{I}$ with $\mathbf{I}$ the $d \times d$ identity matrix and for particular choices of $\theta$, the proposed DG formulation reduces to various well-known interior penalty methods; notable examples include the symmetric and nonsymmetric interior penalty methods of, respectively, Arnold [1] and Rivière *et al.* [25]. Subject to Assumption 1.1, we prove that the proposed DG formulation is well-posed provided the discontinuity penalization parameter is chosen sufficiently large. Moreover, *a priori* error estimates are presented for the error in the broken $H^1(\Omega)$-norm, displaying